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# Quasi-static evolution of delaminated structures: analysis of stability and bifurcation

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## Abstract

Within the framework of dissipative systems with time-independent behavior, the study of the evolution of delaminated structures modeled by frames of plates is considered via a global energetic analysis. Assuming the current equilibrium state is known, the governing rate problem for the instantaneous delamination is formulated as either a system of local equations or as a global variational inequality.

This global formulation enables to study stability and non-bifurcation of the evolution of a delaminated structure under quasi-static loading, corresponding to the statement of existence and uniqueness criteria for the rate solution.

Two analytical applications to simple structures are presented.

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**Keywords:** Delamination; Plates; Kirchhoff–Love theory; Stability; Bifurcation; Propagation

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## 1. Introduction

The study of fracture in composite structures has been widely investigated. The main issues are the nucleation of the damage and its evolution (Moon et al., 2002; La Saponara et al., 2002; Bruno and Greco, 2001; Zou et al., 2002). In the latter category, numerous papers deal with the modeling of delaminated structures where various viewpoints are considered. The delamination can be represented as an imperfect interface whose total damage represents the delamination (Perret et al., 1998; Borg et al., 2002; Greco et al., 2002; Qiu et al., 2001). This assumption is especially efficient to define the start of fracture. Another line of investigation considers the delamination as the propagation of a crack parallel to a plate or a shell which corresponds to the delaminated structure (Storakers and Anderson, 1988; Larson, 1991; Cochebin and Potier-Ferry, 1991; Pradeilles-Duval, 2001; Ousset, 1999). Analytical or numerical examples of propagation of delamination in such composite structures use various propagation criteria mainly based on energy criteria. Extended Griffith formulations, sometimes with decomposition into fracture modes, have been used as well (Hutchinson et al., 2000; Nilsson et al., 2001).

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This paper is devoted to the evolution of delaminated structures represented by a frame of plates. The goal is the introduction of a global formulation for the rate problem, when the propagation is governed by a Griffith criterion. This new formulation enables to analyze the existence and the uniqueness of the solution of this problem. In other words, the present global formulation allows the study of the *stability* and the *bifurcation* in the propagation of the delamination in the modeled delaminated plates, which, to our knowledge, has not been reported in the literature. The approach followed here is an extension of previous work on irreversible phase transformation in three-dimensional bodies (Pradeilles-Duval and Stolz, 1995).

In Section 2, given the kinematic theory chosen in each plate, the kinematic junctions between the plates along the delamination front is defined. Using a classical thermodynamic framework for time-independent behavior of structures (Nguyen, 1993), we derive the equations for the equilibrium problem without propagation (i.e. the delamination front given). Then the value of the total dissipation when the front propagates is obtained. This gives rise to the energy release rate associated to the propagation of the delamination front.

In Section 3, in order to define the quasi-static evolution of the system, one has to set the rate boundary value problem. Having chosen an extended Griffith's law as criterion of delamination and a normality rule as evolution law, the solution, in term of velocity and propagation of the front, is to be governed by either a system of local equations or a global formulation in the form of a variational inequality, based on a potential on the rate mechanical quantities. For a given delamination propagation, the rate boundary value problem corresponds to a non-classical elastic boundary value problem with internal pre-stresses.

In Section 4, it is shown that the evolution of the system (i.e. propagation of front), assuming a classical criterion for the evolution of the front, is governed by a variational inequality. This formulation gives some conclusions on stability and bifurcation of the current state, without the need to actually determine the new geometry obtained by propagation.

Extensions of the above formulation given for Kirchhoff–Love plates are obtained for plates usually attributed to von Karman (i.e. when large transverse displacements and large deformations are considered) in Section 5.

Finally, analytical applications of these formulations to simple structures are proposed in Section 6.

A numerical procedure based on finite element method is outlined in Appendix A.

## 2. Problem settings

### 2.1. Geometry

Using classical modeling of delaminated structures (Cochelin, 1988; Larson, 1991; Storakers, 1991), the system, denoted by  $\Omega_t$ , is considered as three assembled plates, one of which represents the undamaged part ( $\Omega_t^0$ ) and the two other plates correspond to the plates above ( $\Omega_t^1$ ) and below ( $\Omega_t^2$ ) the delamination (see Fig. 1). Consequently,  $\Omega_t = \Omega_t^0 \cup \Omega_t^1 \cup \Omega_t^2$ . The front of delamination along which the three plates are linked, is denoted by  $\Gamma_t$ . The external boundary of  $\Omega_t$  is  $\partial\Omega$ .

The subscript  $t$  emphasizes the fact that the geometry is time-dependent during the evolution of the structure, the sub-domains  $\Omega_t^0$ ,  $\Omega_t^1$  and  $\Omega_t^2$  as well as the delamination front  $\Gamma_t$  are expected to change due to the propagation of the delamination. On the other hand, the external boundary  $\partial\Omega$  is assumed to be unchanged.

The plates are parallel to the Cartesian orthonormal frame  $(\underline{e}_1, \underline{e}_2)$ . Letting  $\underline{e}_3 = \underline{e}_1 \wedge \underline{e}_2$  denote the transverse direction, normal to the initial (undeformed) configuration of the plates, the following notation are introduced:

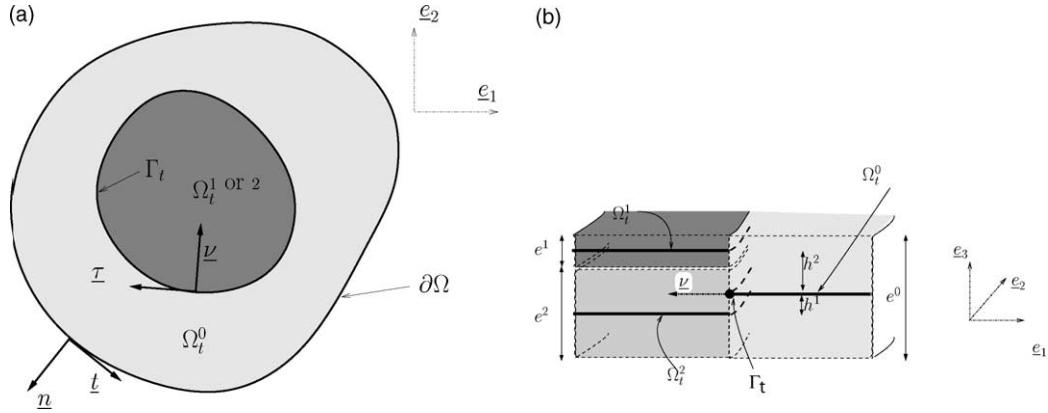


Fig. 1. Delamination modeled by plates frame: (a) top view, (b) transverse view.

- $h^i$ , the vertical position of the plate  $i$  compared with the mean plane of the undamaged plate ( $\Omega_t^0$ ),
- $\underline{n}$ , the unit normal vector to  $\partial\Omega$ , external to  $\Omega_t$  and  $\underline{t} = \underline{e}_3 \wedge \underline{n}$  the tangent vector to  $\partial\Omega$ ,
- $\underline{v}$  and  $\underline{\tau}$ , the unit normal and tangent vectors to  $\Gamma_t$ , with  $\underline{v} \wedge \underline{\tau} = \underline{e}_3$  and  $\underline{v}$  external to the undamaged part  $\Omega_t^0$ .

In addition, a summation convention is used, with a summation range of either 0–2 (Latin superscripts) or 1–2 (Greek subscripts). Latin superscripts refer to the plate  $\Omega_t^0$ ,  $\Omega_t^1$ ,  $\Omega_t^2$ , supporting the corresponding quantity.

$$\sum_{\alpha=1}^2 x_\alpha f_\alpha = x_\alpha f_\alpha = x_1 f_1 + x_2 f_2 \quad \text{and} \quad \sum_{i=0}^2 f^i g^i = f^i g^i = f^0 g^0 + f^1 g^1 + f^2 g^2.$$

Moreover, the plates  $\Omega_t^0$ ,  $\Omega_t^1$  and  $\Omega_t^2$  are parallel but not in the same plane. In the following, they are considered through their projection on the mean plane of  $\Omega_t^0$ . So, a point is defined by its horizontal position  $(x_1, x_2)$  and by the plane it bellows which indicates its vertical position  $(x_3 = h^i)$ . One should note that, as shown in Fig. 1, a position  $(x_1, x_2)$  corresponds to two points, one in  $\Omega_t^1$  and another in  $\Omega_t^2$ .

Consequently,  $\Gamma_t$  corresponds to the curve in the mean plane of  $\Omega_t^0$ .

In the following,  $\nabla f$  defines the gradient of  $f$ , in the plane  $(\underline{e}_1, \underline{e}_2)$ . Two different kinds of jumps across  $\Gamma_t$  are defined:  $\llbracket f \rrbracket = f^0 - f^1 - f^2$  and  $\llbracket f \rrbracket^i = f^0 - f^i$ . The gradient of quantity  $f$  in the direction  $\beta$  is denoted by  $f_{,\beta} = \nabla f \cdot \underline{e}_\beta$ . If  $s$  denotes the curvilinear coordinate along  $\Gamma_t$ , associated to  $\underline{\tau}$ , then, for instance,  $f_{,v} = \nabla f \cdot \underline{v}$  and  $f_{,s} = \nabla f \cdot \underline{\tau}$ .

## 2.2. Kinematic description

The motion in the plate  $i$  is classically defined by

- the in-plane displacement,  $\underline{u}^i(x_1, x_2) = u_\alpha^i(x_1, x_2)\underline{e}_\alpha$ ,
- the transverse displacement of the mean plane,  $w^i(x_1, x_2)\underline{e}_3$ ,
- the rotation of a small segment initially normal to the plate, in other words, the flexural rotation,  $\underline{\theta}^i(x_1, x_2) = \theta_\alpha^i(x_1, x_2)\underline{e}_\alpha$ .

So, the displacement  $\underline{\xi}^i$  of the particle whose initial coordinates are  $(x_1, x_2, x_3)$  is:

$$\underline{\xi}^i(x_1, x_2, x_3) = \underline{u}^i(x_1, x_2) + w^i(x_1, x_2)\underline{e}_3 - x_3 \underline{\theta}^i(x_1, x_2).$$

The global strain tensor associated to this motion within the framework of small perturbations is given by:

$$\underline{\epsilon}^i(\underline{\underline{\epsilon}}^i) = \underline{\epsilon}^i(\underline{u}^i) + \frac{1}{2}[\underline{e}_3 \otimes \underline{\gamma}^i(w^i, \underline{\theta}^i) + \underline{\gamma}^i(w^i, \underline{\theta}^i) \otimes \underline{e}_3] - x_3 \underline{\kappa}^i(\underline{\theta}^i),$$

with

$$\underline{\epsilon}(\underline{u}) = \frac{1}{2}(\nabla \underline{u} + {}^T \nabla \underline{u}), \quad \underline{\kappa}(\underline{\theta}) = \frac{1}{2}(\nabla \underline{\theta} + {}^T \nabla \underline{\theta}), \quad \underline{\gamma}(w, \underline{\theta}) = \nabla w - \underline{\theta}. \quad (1)$$

$\underline{\epsilon}$ ,  $\underline{\kappa}$  and  $\underline{\gamma}$  are respectively the plane strain of the mean plane, the gradient of the rotation of the normal and the distortion of the plate.

Along the delamination front, the motions of the three plates are linked. One can choose among several sets of conditions to formulate this link and this choice can strongly influence the predicted behavior of the whole structure (Anquez et al., 1990; Pradeilles-Duval, 1992; Roudolff and Ousset, 2002). In this paper, the continuity of plane displacement, of transverse displacement and of rotation along  $\Gamma$ , are enforced through the following conditions:

$$\left. \begin{array}{l} \llbracket \underline{u} \rrbracket^i - h^i \underline{\theta}^0 = \underline{u}^0 - h^i \underline{\theta}^0 - \underline{u}^i = \underline{0} \\ \llbracket w \rrbracket^i = w^0 - w^i = 0 \\ \llbracket \underline{\theta} \rrbracket^i = \underline{\theta}^0 - \underline{\theta}^i = \underline{0} \end{array} \right\} \quad \forall i \in \{1, 2\}. \quad (2)$$

These conditions imply the following relations concerning the tangential derivative of the displacement:

$$\left. \begin{array}{l} \llbracket \nabla \underline{u} \rrbracket^i \cdot \underline{\tau} - h^i \nabla \underline{\theta}^0 \cdot \underline{\tau} = \underline{0} \\ \llbracket \nabla w \rrbracket^i \cdot \underline{\tau} = 0 \\ \llbracket \nabla \underline{\theta} \rrbracket^i \cdot \underline{\tau} = \underline{0} \end{array} \right\} \quad \forall i \in \{1, 2\}.$$

### 2.3. Constitutive relations

Each plate is assumed to be elastic and the elastic energy is denoted by  $W$ , which is a function of  $\underline{\epsilon}(\underline{u})$ ,  $\underline{\kappa}(\underline{\theta})$ ,  $\underline{\gamma}(w, \underline{\theta})$ .

So, in the following, the generalized stresses are the in-plane stress tensor,  $\mathbf{N}$ , the bending stress tensor,  $\mathbf{M}$ , and the shear force,  $\underline{\mathbf{T}}$ . They are obtained in each plate through the constitutive relations:

$$\mathbf{N} = \frac{\partial W}{\partial \underline{\epsilon}}, \quad \mathbf{M} = \frac{\partial W}{\partial \underline{\kappa}}, \quad \underline{\mathbf{T}} = \frac{\partial W}{\partial \underline{\gamma}}. \quad (3)$$

### 2.4. Equilibrium

The system of plates is loaded by prescribed displacements and forces. Classically, the latter are represented by vector densities over the external boundary, assuming that no surface force is applied. Then, if the motion of the structure is defined by the generalized velocities  $(\underline{u}^*, w^*, \underline{\theta}^*)$ , the virtual power of external forces is reduced to:

$$\mathcal{P}_e(\underline{u}^*, w^*, \underline{\theta}^*) = \int_{\partial\Omega} [\underline{F} \cdot (\underline{u}^* + w^* \underline{e}_3) + \underline{C} \cdot \underline{\theta}^*] \, ds.$$

In the previous equation,  $\underline{F}$  denotes the external distributed force and  $\underline{C}$  the external distributed moment on  $\partial\Omega$ .

Similarly, according to the previous definition of constitutive relations, the virtual power of internal forces is:

$$\mathcal{P}_i(\underline{u}^*, w^*, \underline{\theta}^*) = - \int_{\Omega_t} [\mathbf{N} : \boldsymbol{\varepsilon}(\underline{u}^*) + \mathbf{M} : \boldsymbol{\kappa}(\underline{\theta}^*) + \underline{T} \cdot \underline{\gamma}(w^*, \underline{\theta}^*)] \, d\omega.$$

By virtue of the virtual power principle, a state of equilibrium is governed by  $\mathcal{P}_i(\underline{u}^*, w^*, \underline{\theta}^*) + \mathcal{P}_e(\underline{u}^*, w^*, \underline{\theta}^*) = 0$ , for all fields  $(\underline{u}^*, w^*, \underline{\theta}^*)$  which satisfy the continuity relations (2). This leads to the classical local equilibrium equations:

$$\left. \begin{array}{l} \operatorname{div} \mathbf{N} = 0 \\ \operatorname{div} \mathbf{M} + \underline{T} = 0 \\ \operatorname{div} \underline{T} = 0 \end{array} \right\} \text{ in } \Omega_t^i, \quad \text{with } i \in \{0, 1, 2\}, \quad (4)$$

and the relations between generalized stresses and distributed forces on the boundary  $\partial\Omega$  are

$$\mathbf{N} \cdot \underline{n} = F_\beta e_\beta, \quad \underline{T} \cdot \underline{n} = F_3, \quad \mathbf{M} \cdot \underline{n} = \underline{C}.$$

Moreover, the continuity relations (2) along  $\Gamma_t$  induce static junctions:

$$[\mathbf{N}] \cdot \underline{y} = \underline{0}, \quad [\underline{T}] \cdot \underline{y} = \underline{0}, \quad [\mathbf{M}] \cdot \underline{y} + h^i \mathbf{N}^i \cdot \underline{y} = \underline{0}.$$

## 2.5. Equations for Kirchhoff–Love plates

The main results of this article are established for Kirchhoff–Love plates (Timoshenko and Woinwsky-Krieger, 1959). Their extension to von Karman plates (Timoshenko and Woinwsky-Krieger, 1959, Chapter 13; Love, 1944) is then addressed in Section 5.

In the framework of Kirchhoff–Love plates, one has  $\nabla w = \underline{\theta}$ , which corresponds to  $\underline{\gamma}(w, \underline{\theta}) = \underline{0}$  and  $\boldsymbol{\kappa} = \nabla \nabla w$ . The equilibrium equations (4) reduce to:

$$\left. \begin{array}{l} \operatorname{div} \mathbf{N} = 0 \\ \operatorname{div} \operatorname{div} \mathbf{M} = 0 \end{array} \right\} \text{ in } \Omega_t^i, \quad \text{with } i \in \{0, 1, 2\}. \quad (5)$$

Thanks to relations between  $w$  and  $\underline{\theta}$  along  $\partial\Omega$  and  $\Gamma_t$ , the boundary conditions on  $\partial\Omega$  become:

$$\begin{aligned} \mathbf{N} \cdot \underline{n} &= F_\beta e_\beta, \\ -\operatorname{div} \mathbf{M} \cdot \underline{n} - \frac{\partial}{\partial s} (\underline{n} \cdot \mathbf{M} \cdot \underline{t}) &= F_3 - \frac{\partial C_s}{\partial s}, \\ \underline{n} \cdot \mathbf{M} \cdot \underline{n} &= C_n \end{aligned} \quad (6)$$

and the continuity conditions on  $\Gamma_t$  become:

$$\begin{aligned} [\mathbf{N}] \cdot \underline{y} &= \underline{0}, \\ [\operatorname{div} \mathbf{M}] \cdot \underline{y} + \frac{\partial}{\partial s} [\underline{y} \cdot \{[\mathbf{M}] + h^i \mathbf{N}^i\} \cdot \underline{t}] &= 0, \\ \underline{y} \cdot [\mathbf{M}] \cdot \underline{y} + h^i \underline{y} \cdot \mathbf{N}^i \cdot \underline{y} &= 0. \end{aligned} \quad (7)$$

Define the following notations on any line in  $\Omega_t$  ( $\underline{y}$  is a normal vector to this line):

- $\underline{\tilde{M}}^i = \mathbf{M}^i \cdot \underline{y} - h^i \mathbf{N}^i \cdot \underline{y}$ , the moment on the line,
- $Q^i = \underline{T}^i \cdot \underline{y} - \frac{\partial}{\partial s} (\underline{\tilde{M}}^i \cdot \underline{t})$ , the transverse force on the line.

Then, the boundary conditions on  $\partial\Omega$  and the continuity relations on  $\Gamma_t$  are rewritten as:

- on  $\partial\Omega$ ,

$$\mathbf{N} \cdot \underline{n} = F_\beta e_\beta, \quad Q = F_3 - \frac{\partial C_s}{\partial s}, \quad \underline{\tilde{M}} \cdot \underline{n} = C_n, \quad (8)$$

- on  $\Gamma_t$ ,

$$[\![\mathbf{N}]\!] \cdot \underline{y} = 0, \quad [\![\mathbf{Q}]\!] = 0, \quad [\![\tilde{\mathbf{M}}]\!] \cdot \underline{y} = 0. \quad (9)$$

## 2.6. Prescribed loading and displacement

The virtual work associated to given external forces (resp. prescribed displacements) is denoted by  $\lambda \mathcal{P}_{\text{ext}}(\underline{u}, w)$  (resp.  $\lambda \mathcal{P}_{\text{kin}}(\mathbf{N}, \mathbf{M})$ ) and is written as

$$\mathcal{P}_{\text{ext}}(\underline{u}, w) = \sum_{k=1}^3 \int_{\partial\Omega_{F_k}} F_k^d [\underline{e}_k \cdot (\underline{u} + w \underline{e}_3)] \, ds + \int_{\partial\Omega_C} C^d (\underline{n} \cdot \nabla w) \, ds \quad (10)$$

and

$$\mathcal{P}_{\text{kin}}(\mathbf{N}, \mathbf{M}) = \sum_{\beta=1}^2 \int_{\partial\Omega_{u_\beta}} u_\beta^d [\underline{e}_\beta \cdot (\mathbf{N} \cdot \underline{n})] \, ds - \int_{\partial\Omega_w} \left[ \text{div} \mathbf{M} \cdot \underline{n} + \frac{\partial}{\partial s} \cdot (\underline{n} \cdot \mathbf{M} \cdot \underline{t}) \right] w^d \, ds + \int_{\partial\Omega_\theta} (\underline{n} \cdot \tilde{\mathbf{M}}) \theta^d \, ds, \quad (11)$$

where  $\partial\Omega_f$  denotes the part of the total external boundary  $\partial\Omega$  where the quantity  $f$  is prescribed.

In the previous formula,  $\lambda$  represents the loading parameter (i.e. the prescribed data on the boundary at time  $t$  are  $\lambda(t)F_k^d(x_1, x_2)$  on  $\partial\Omega_{F_k}$ ,  $\lambda(t)C^d(x_1, x_2)$  on  $\partial\Omega_C$ ,  $\lambda(t)u_\beta^d(x_1, x_2)$  on  $\partial\Omega_{u_\beta}$ ,  $\lambda(t)w^d(x_1, x_2)$  on  $\partial\Omega_w$ ,  $\lambda(t)\theta^d(x_1, x_2)$  on  $\partial\Omega_\theta$ ). The evolution of the structure is studied when this parameter increases.

Prescribed loading and generalized displacements are given so that the problem is “well-posed”:

- $\partial\Omega = \partial\Omega_{u_\beta} \cup \partial\Omega_{F_\beta}$  and  $\partial\Omega_{u_\beta} \cap \partial\Omega_{F_\beta} = \emptyset \forall \beta \in \{1, 2\}$  (complementarity of the prescribed in-plane displacement and the associated in-plane force),
- $\partial\Omega = \partial\Omega_w \cup \partial\Omega_{F_3}$  and  $\partial\Omega_w \cap \partial\Omega_{F_3} = \emptyset$  (complementarity of the transverse displacement and the shear force),
- $\partial\Omega = \partial\Omega_\theta \cup \partial\Omega_C$  and  $\partial\Omega_\theta \cap \partial\Omega_C = \emptyset$  (complementarity of the flexural rotation and the normal bending moment).

## 2.7. Equilibrium equations

The total potential energy of the system is

$$E^{\text{pot}}(\underline{u}, w, \Gamma_t, \lambda) = \int_{\Omega_t} W(\underline{u}, w) \, d\omega - \lambda \mathcal{P}_{\text{ext}}(\underline{u}, w).$$

Using standard Lagrangian method, with the kinematic constraints defined by (2), the following system governs equilibrium for a given front  $\Gamma_t$ :

- boundary conditions with prescribed displacement:

$$\begin{aligned} \forall \beta \in \{1, 2\}, \quad u_\beta &= \lambda u_\beta^d \quad \text{on } \partial\Omega_{u_\beta}, \\ w &= \lambda w^d \quad \text{on } \partial\Omega_w, \\ \nabla w \cdot \underline{n} &= \lambda \theta_n^d \quad \text{on } \partial\Omega_{\theta_n}, \end{aligned} \quad (12)$$

- compatibility relations (1) on the plates,
- compatibility relations (2) along the delamination front  $\Gamma_t$  through kinematic junction,
- constitutive law (3) on each plate,

- classical local equilibrium equations (5) on generalized stresses,
- equilibrium equations (7) or (9) due to the junction along  $\Gamma_t$ ,
- boundary conditions with given external loading:

$$\begin{aligned} \forall \beta \in \{1, 2\}, \quad \underline{\varrho}_\beta \cdot \underline{N} \cdot \underline{n} &= \lambda F_\beta^d \quad \text{on } \partial\Omega_{F_\beta}, \\ \underline{Q} &= \lambda F_3^d \quad \text{on } \partial\Omega_{F_3}, \\ \underline{\tilde{M}} \cdot \underline{n} &= \lambda C^d \quad \text{on } \partial\Omega_C. \end{aligned} \quad (13)$$

The displacement, solution of this boundary value problem,  $(\underline{u}(\lambda, \Gamma_t), w(\lambda, \Gamma_t))$  is also characterized as the kinematically admissible minimizer of the potential energy. The equilibrium value of  $E^{\text{pot}}$  will play an important role in the analysis to follow and will be denoted by  $P$ , i.e.  $P(\lambda, \Gamma_t) = E^{\text{pot}}(\underline{u}(\lambda, \Gamma_t), w(\lambda, \Gamma_t), \lambda)$ .

## 2.8. Propagation of the delamination: associated derivative

When the loading parameter increases, the delamination is expected to propagate. Let  $-\phi \underline{v}$  denote the normal velocity of the front  $\Gamma_t$  in the mean plane of plate  $\Omega_t^0$ , with  $\phi(s) \geq 0, \forall s \in \Gamma_t$ . So that, if  $\underline{x}$  on  $\Gamma_t$ , then  $\underline{x} - \phi \underline{v} dt$  is on  $\Gamma_{t+dt}$ .

For any quantity  $f$  defined along  $\Gamma_t$ , the time derivative  $D_\phi(f)$  associated to the propagation is defined as:

$$D_\phi(f)(\underline{x}) = \lim_{dt \rightarrow 0} \frac{f(\underline{x} - \phi \underline{v} dt, t + dt) - f(\underline{x}, t)}{dt}, \quad (14)$$

where  $\underline{x}$  is a point on  $\Gamma_t$ . In particular, one finds that:

$$D_\phi \underline{v} = \frac{\partial \phi}{\partial s} \underline{\tau} = \phi_{,s} \underline{\tau} \quad \text{and} \quad D_\phi \underline{\tau} = -\frac{\partial \phi}{\partial s} \underline{v} = -\phi_{,s} \underline{v}.$$

For a function  $f$  defined over  $\Omega_t$ , one gets for the value of  $f$  on  $\Gamma_t$ ,

$$D_\phi(f) = \dot{f} - \phi \nabla f \cdot \underline{v},$$

where  $\dot{f}$  is the partial time derivative of  $f$ , i.e.  $\frac{\partial f}{\partial t}$ . Besides, the time derivative of a generic integral over  $\Omega_t$  is given by:

$$\frac{d}{dt} \int_{\Omega_t} f d\omega = \int_{\Omega_t} \dot{f} d\omega - \int_{\Gamma_t} [\![f]\!] \phi ds.$$

During any quasi-static evolution of  $\Gamma_t$  with velocity  $\phi$ , the relations (2) must be hold at any time. So, Hadamard jump relations on displacement are verified:

$$\left. \begin{aligned} D_\phi([\![\underline{u}]\!]^i - h^i \nabla w^0) &= [\![\dot{\underline{u}}]\!]^i - h^i \nabla \dot{w}^0 - \phi \{ [\![\nabla \underline{u}]\!]^i - h^i \nabla \nabla w^0 \} \cdot \underline{v} = \underline{0} \\ D_\phi([\![w]\!]^i) &= [\![\dot{w}]\!]^i = 0 \\ D_\phi([\![\nabla w]\!]^i) &= [\![\nabla \dot{w}]\!]^i - \phi [\![\nabla \nabla w]\!]^i \cdot \underline{v} = \underline{0} \end{aligned} \right\} \quad \forall i \in \{1, 2\}. \quad (15)$$

Likewise, enforcement of the static continuity relations (7) at any time implies:

$$\begin{aligned} \llbracket \dot{\mathbf{N}} \rrbracket \cdot \underline{v} &= -\frac{\partial}{\partial s}(\phi \llbracket \mathbf{N} \rrbracket \cdot \underline{\iota}), \\ \llbracket \operatorname{div} \dot{\mathbf{M}} \rrbracket \cdot \underline{v} + \frac{\partial}{\partial s} \left\{ \underline{v} \cdot (\llbracket \dot{\mathbf{M}} \rrbracket + h^i \dot{\mathbf{N}}^i) \cdot \underline{\iota} \right\} &= -\frac{\partial}{\partial s} \left( \underline{\iota} \cdot \frac{\partial}{\partial s} \left\{ \phi (\llbracket \mathbf{M} \rrbracket + h^i \mathbf{N}^i) \right\} \cdot \underline{\iota} \right), \\ \underline{v} \cdot (\llbracket \dot{\mathbf{M}} \rrbracket + h^i \dot{\mathbf{N}}^i) \cdot \underline{v} &= \phi [\underline{v} \cdot \{ \nabla (\llbracket \mathbf{M} \rrbracket + h^i \mathbf{N}^i) \cdot \underline{v} \} \cdot \underline{v}] - (\llbracket \mathbf{M} \rrbracket + h^i \mathbf{N}^i) : (\underline{\iota} \otimes \underline{v} + \underline{v} \otimes \underline{\iota}) \frac{\partial \phi}{\partial s}, \end{aligned} \quad (16)$$

or

$$\begin{aligned} \llbracket \dot{\mathbf{N}} \rrbracket \cdot \underline{v} &= -\frac{\partial}{\partial s}(\phi \llbracket \mathbf{N} \rrbracket \cdot \underline{\iota}), \\ \llbracket \dot{\mathbf{Q}} \rrbracket &= -\frac{\partial}{\partial s} \left( \underline{\iota} \cdot \frac{\partial}{\partial s} \left\{ \phi (\llbracket \mathbf{M} \rrbracket + h^i \mathbf{N}^i) \right\} \cdot \underline{\iota} \right), \\ \llbracket \dot{\tilde{\mathbf{M}}} \rrbracket \cdot \underline{v} &= \phi [\underline{v} \cdot \{ \nabla (\llbracket \mathbf{M} \rrbracket + h^i \mathbf{N}^i) \cdot \underline{v} \} \cdot \underline{v}] - \frac{\partial \phi}{\partial s} (\llbracket \tilde{\mathbf{M}} \rrbracket \cdot \underline{\iota}). \end{aligned}$$

## 2.9. Total dissipation

When the loading parameter  $\lambda$  increases or the delamination front  $\Gamma_t$  propagates, the evolution along the equilibrium path of the potential energy at equilibrium  $P$  is given by:

$$\begin{aligned} \frac{dP}{dt} &= \frac{d}{dt} [E^{\text{pot}}(\underline{u}(\Gamma_t, \lambda), w(\Gamma_t, \lambda), \Gamma_t, \lambda)] = \frac{d}{dt} \left[ \int_{\Omega_t} W(\underline{u}(\Gamma_t, \lambda), w(\Gamma_t, \lambda)) \, d\omega - \lambda \mathcal{P}_{\text{ext}}(\underline{u}(\Gamma_t, \lambda), w(\Gamma_t, \lambda)) \right] \\ &= E_{,\underline{u}}^{\text{pot}} \frac{d}{dt} \underline{u} + E_{,w}^{\text{pot}} \frac{d}{dt} w + E_{,\Gamma_t}^{\text{pot}} \frac{d}{dt} \Gamma_t + E_{,\lambda}^{\text{pot}} \dot{\lambda}. \end{aligned}$$

Thanks to the behavior relation (3), to the continuity equation on  $\Gamma_t$  (7) and to virtual power principle,

$$\begin{aligned} &\int_{\Omega_t} \mathbf{N} : \nabla \delta \underline{u} + \mathbf{M} : \nabla \nabla \delta w \, da \\ &= \int_{\partial \Omega} \left\{ \delta \underline{u} \cdot (\mathbf{N} \cdot \underline{n}) - \left[ \operatorname{div} \mathbf{M} \cdot \underline{n} + \frac{\partial}{\partial s} (\underline{n} \cdot \mathbf{M} \cdot \underline{\iota}) \right] \delta w + (\underline{n} \cdot \mathbf{M} \cdot \underline{n}) (\nabla \delta w \cdot \underline{n}) \right\} \, ds \\ &\quad - \int_{\Gamma_t} \{ \underline{v} \cdot \mathbf{N}^i \cdot (\llbracket \delta \underline{u} \rrbracket^i - h^i \delta \nabla w^0) + \underline{v} \cdot \mathbf{M}^i \cdot \llbracket \nabla \delta w \rrbracket^i - (\operatorname{div} \mathbf{M}^i \cdot \underline{v}) \llbracket \delta w \rrbracket^i \} \, ds \end{aligned} \quad (17)$$

thus, using the Hadamard jump relations (15), ones obtains:

$$\begin{aligned} E_{,\underline{u}}^{\text{pot}} \frac{d}{dt} \underline{u}(\Gamma_t, \lambda) + E_{,w}^{\text{pot}} \frac{d}{dt} w(\Gamma_t, \lambda) &= \int_{\Omega_t} \mathbf{N} : \nabla \dot{\underline{u}} + \mathbf{M} : \nabla \nabla \dot{w} \, da - \lambda \mathcal{P}_{\text{ext}}(\dot{\underline{u}}, \dot{w}) \\ &= \dot{\lambda} \mathcal{P}_{\text{kin}}(\mathbf{N}, \mathbf{M}) \\ &\quad + \int_{\Gamma_t} \{ \underline{v} \cdot \mathbf{N}^i \cdot (\llbracket \nabla \underline{u} \rrbracket^i - h^i \nabla \nabla w^0) + \underline{v} \cdot \mathbf{M}^i \cdot \llbracket \nabla \nabla w \rrbracket^i \} \cdot \underline{v} \phi \, ds. \end{aligned} \quad (18)$$

The total time derivative of the potential energy at equilibrium is thus found to be given by:

$$\begin{aligned} \frac{dP}{dt} &= \dot{\lambda} \mathcal{P}_{\text{kin}}(\mathbf{N}, \mathbf{M}) - \dot{\lambda} \mathcal{P}_{\text{ext}}(\underline{u}, w) - \int_{\Gamma_t} \llbracket W \rrbracket \phi \, ds + \int_{\Gamma_t} \{ \underline{v} \cdot \mathbf{N}^i \cdot (\llbracket \nabla \underline{u} \rrbracket^i - h^i \nabla \nabla w^0) + \underline{v} \cdot \mathbf{M}^i \cdot \llbracket \nabla \nabla w \rrbracket^i \} \cdot \underline{v} \phi \, ds. \end{aligned} \quad (19)$$

The total dissipated power of the system  $\mathcal{D}$  is given by the difference between the power of external forces and the rate of total free energy of the system, i.e. by:

$$\begin{aligned}\mathcal{D} &= \int_{\partial\Omega} [\underline{F} \cdot (\dot{\underline{u}} + \dot{\underline{w}}\underline{e}_3) + C\nabla\dot{\underline{w}} \cdot \underline{n}] \, ds - \frac{d}{dt} \int_{\Omega_t} W(\underline{u}, \underline{w}) \, d\omega = -\frac{dP}{dt} - \dot{\lambda}\mathcal{P}_{\text{ext}}(\underline{u}, \underline{w}) + \dot{\lambda}\mathcal{P}_{\text{kin}}(\underline{N}, \underline{M}) \\ &= -P_{\Gamma_t} \frac{d\Gamma_t}{dt}.\end{aligned}\quad (20)$$

The total dissipated power is thus found, by virtue of (19), to be of the form:

$$\mathcal{D} = \int_{\Gamma_t} G(s)\phi(s) \, ds, \quad (21)$$

where the energy release rate  $G(s)$  is given by:

$$G = \llbracket W \rrbracket - \underline{y} \cdot \underline{N}_i \cdot (\llbracket \nabla \underline{u} \rrbracket^i - h_i \nabla \nabla w_0) \cdot \underline{y} - \underline{y} \cdot \underline{M}_i \cdot \llbracket \nabla \nabla w \rrbracket^i \cdot \underline{y}. \quad (22)$$

### 3. Quasi-static evolution

In this section, the quasi-static evolution is presented. It includes the definition of the criterion which governs the propagation of the delamination, then the local equations as well as the global formulation of the rate boundary value problem are presented.

#### 3.1. Propagation criterion

Taking into account the form of the dissipation, the governing law for the evolution of  $\Gamma_t$  is introduced according to the framework of the generalized standard materials (Nguyen, 1987, 1993).

Accordingly, the existence of a convex threshold function  $g(G)$  is postulated, so that the propagation of the delamination front is subject to the following law:

- if  $g(G(s)) < 0$ , then  $\phi(s) = 0$ , i.e. no propagation at point  $s$ ,
- if  $g(G(s)) = 0$ , then  $\phi(s) \geq 0$ , i.e. the propagation of  $\Gamma_t$  at point  $s$  is possible,

which is equivalent to

$$\phi(s) = c(s) \frac{\partial g}{\partial G} \quad \text{with } c(s)g(G(s)) = 0 \text{ (consistency relation) and } c(s) \geq 0. \quad (23)$$

A generalized Griffith criterion can be considered by introducing a threshold energy to be reached for propagation of the delamination front to occur. In this case, one would have  $g(G) = G - G_C$ .

In the following,  $\Gamma_t^{\text{rupt}}$  is the subset of the front  $\Gamma_t$  where the criterion is reached and the propagation is possible.

$$\Gamma_t^{\text{rupt}} = \{s \in \Gamma_t \text{ such that } g(G(s)) = 0\}.$$

#### 3.2. Consistency condition and evolution law

When the front is moving, Eq. (23) implies that, during the actual propagation of  $\Gamma_t^{\text{rupt}}$ ,  $g(G(s)) = 0$  must be maintained, leading to the consistency equation:

$$\text{if } c > 0 \text{ on } \Gamma_t^{\text{rupt}}, \quad D_\phi(g(G)) = \frac{\partial g}{\partial G} D_\phi(G) = 0.$$

In the present formulation,

$$D_\phi(G) = \tilde{G}(\dot{\mathbf{N}}, \dot{\mathbf{M}}) + \hat{G}(\dot{\mathbf{u}}, \dot{\mathbf{w}}) + \phi G^*, \quad (24)$$

where

$$\begin{aligned} \tilde{G}(\dot{\mathbf{N}}, \dot{\mathbf{M}}) &= -(\underline{\mathbf{v}} \cdot \dot{\mathbf{N}}_i) \cdot (\llbracket \nabla \underline{\mathbf{u}} \rrbracket^i - h_i \boldsymbol{\kappa}_0) \cdot \underline{\mathbf{v}} - (\underline{\mathbf{v}} \cdot \dot{\mathbf{M}}_i) \cdot (\llbracket \boldsymbol{\kappa} \rrbracket^i \cdot \underline{\mathbf{v}}), \\ \hat{G}(\dot{\mathbf{u}}, \dot{\mathbf{w}}) &= -\llbracket \mathbf{N} \rrbracket \cdots \nabla \dot{\mathbf{u}}_0 + (\llbracket \mathbf{M} \rrbracket + h_i \mathbf{N}_i) \cdots (\dot{\boldsymbol{\kappa}}_0), \\ G^* &= -\llbracket \mathbf{M} \cdots \nabla \nabla \underline{\mathbf{u}} + \mathbf{M} \cdots \nabla \nabla \nabla \mathbf{w} \rrbracket \cdot \underline{\mathbf{v}} + (\underline{\mathbf{v}} \cdot \nabla \mathbf{N}_i \cdot \underline{\mathbf{v}}) \cdot (\llbracket \nabla \underline{\mathbf{u}} \rrbracket^i - h_i \boldsymbol{\kappa}_0) \cdot \underline{\mathbf{v}} \\ &\quad + (\underline{\mathbf{v}} \cdot \nabla \mathbf{M}_i \cdot \underline{\mathbf{v}}) \cdot \llbracket \boldsymbol{\kappa} \rrbracket^i \cdot \underline{\mathbf{v}} + \underline{\mathbf{z}} \cdot \mathbf{N}_i \cdot [(\llbracket \nabla \nabla \underline{\mathbf{u}} \rrbracket^i - h_i \nabla \boldsymbol{\kappa}_0) \cdot \underline{\mathbf{v}}] \cdot \underline{\mathbf{z}} + \underline{\mathbf{z}} \cdot \mathbf{M}_i \cdot (\llbracket \nabla \boldsymbol{\kappa} \rrbracket^i \cdot \underline{\mathbf{v}}) \cdot \underline{\mathbf{z}}. \end{aligned}$$

Finally, the evolution law of the delamination front is:

$$c \in V / \forall c' \in V, \quad D_\phi(G)(c' - c) \frac{\partial g}{\partial G} \leq 0 \quad (25)$$

with

$$V = \{c(s) \text{ with } c \geq 0 \text{ on } \Gamma_t^{\text{rupt}} \text{ and } c = 0 \text{ on } \Gamma_t \setminus \Gamma_t^{\text{rupt}}\}. \quad (26)$$

### 3.3. Local equations of the rate problem

The evolution law for  $\Gamma_t$  being chosen, the instantaneous evolution of the global system induced by a loading evolution with velocity given by  $\dot{\lambda}$  is now investigated. This evolution turns out to be governed by the following rate problem for the rates of fields variables  $(\dot{\mathbf{u}}, \dot{\mathbf{w}})$ :

- boundary conditions with prescribed displacements:

$$\begin{aligned} \forall \beta \in \{1, 2\}, \quad \dot{\mathbf{u}}_\beta &= \dot{\lambda} u_\beta^d \quad \text{on } \partial \Omega_{u_\beta}, \\ \dot{\mathbf{w}} &= \dot{\lambda} w^d \quad \text{on } \partial \Omega_w, \\ \nabla \dot{\mathbf{w}} \cdot \underline{\mathbf{n}} &= \dot{\lambda} \theta_n^d \quad \text{on } \partial \Omega_{\theta_n}, \end{aligned} \quad (27)$$

- compatibility:

◦ in each plate  $\Omega_t^i$ :

$$\dot{\boldsymbol{\epsilon}}^i = \frac{1}{2} (\nabla \dot{\mathbf{u}}^i + {}^T \nabla \dot{\mathbf{u}}^i), \quad \dot{\boldsymbol{\kappa}}^i = \nabla \nabla \dot{\mathbf{w}}^i, \quad (28)$$

◦ on  $\Gamma_t$  verification of the Hadamard relations (15),

- constitutive relations:

$$\begin{bmatrix} \dot{\mathbf{N}}^i \\ \dot{\mathbf{M}}^i \end{bmatrix} = \begin{pmatrix} \frac{\partial^2 W^i}{\partial \boldsymbol{\epsilon}^2} & \frac{\partial^2 W^i}{\partial \boldsymbol{\epsilon} \partial \boldsymbol{\kappa}} \\ \frac{\partial^2 W^i}{\partial \boldsymbol{\kappa} \partial \boldsymbol{\epsilon}} & \frac{\partial^2 W^i}{\partial \boldsymbol{\kappa}^2} \end{pmatrix} \begin{bmatrix} \dot{\boldsymbol{\epsilon}}^i \\ \dot{\boldsymbol{\kappa}}^i \end{bmatrix} = W^{i''} \begin{bmatrix} \dot{\boldsymbol{\epsilon}}^i \\ \dot{\boldsymbol{\kappa}}^i \end{bmatrix} \quad \text{in } \Omega_t^i, \quad \text{with } i \in \{0, 1, 2\}, \quad (29)$$

- local equilibrium equations for the generalized stresses:

$$\left. \begin{aligned} \text{div } \dot{\mathbf{N}} &= 0 \\ \text{div div } \dot{\mathbf{M}} &= 0 \end{aligned} \right\} \quad \text{in } \Omega_t^i, \quad \text{with } i \in \{0, 1, 2\}, \quad (30)$$

- boundary conditions with prescribed forces:

$$\begin{aligned} \forall \beta \in \{1, 2\}, \quad \underline{\varepsilon}_\beta \cdot \dot{\underline{N}} \cdot \underline{n} &= \dot{\lambda} F_\beta^d \quad \text{on } \partial\Omega_{F_\beta}, \\ \dot{Q} &= -\operatorname{div} \dot{\underline{M}} \cdot \underline{n} - \frac{\partial}{\partial s} (\underline{n} \cdot \dot{\underline{M}} \cdot \underline{t}) = \dot{\lambda} F_3^d \quad \text{on } \partial\Omega_{F_3}, \\ \dot{\underline{C}} \cdot \underline{n} &= \underline{n} \cdot \dot{\underline{M}} \cdot \underline{n} = \dot{\lambda} C^d \quad \text{on } \partial\Omega_C, \end{aligned} \quad (31)$$

- equilibrium conditions in rate form (16) along the moving delamination front  $\Gamma_t$ ,
- evolution law of  $\Gamma_t$  given by (25).

The rate boundary value problem (15), (16) and (27)–(31) on  $\dot{\underline{u}}$  and  $\dot{w}$  is similar to the original equilibrium system (1)–(3), (5), (7), (12) and (13), but, here the right-hand sides of equations depend on a prescribed evolution of the delamination front. The rate boundary value problem with a prescribed evolution of  $\Gamma_t$  (i.e.  $\phi$  or  $c$  given) is similar to a problem of elasticity with non-classical conditions (15) and (16) localized on  $\Gamma_t$ .

### 3.4. Global formulation of the rate value boundary problem

In order to have a formulation better suited to the study of stability and bifurcation, a global functional is defined:

$$F(\dot{\underline{u}}, \dot{w}, c, \dot{\lambda}) = \sum_{i=0}^2 \int_{\Omega_t^i} \frac{1}{2} [\dot{\varepsilon}, \dot{\kappa}] W'' \begin{bmatrix} \dot{\varepsilon} \\ \dot{\kappa} \end{bmatrix} d\omega - \dot{\lambda} \mathcal{P}_{\text{ext}}(\dot{\underline{u}}, \dot{w}) - \int_{\Gamma_t} \widehat{G}(\dot{\underline{u}}, \dot{w}) c \frac{\partial g}{\partial G} ds - \frac{1}{2} \int_{\Gamma_t} G^* c^2 \left( \frac{\partial g}{\partial G} \right)^2 ds, \quad (32)$$

where  $\dot{\varepsilon}$  and  $\dot{\kappa}$  are defined by (28).

**Theorem 1.**  $(\dot{\underline{u}}, \dot{w}, c)$  is the solution of the system of local equations (15), (16), (25), (27), (30), (31) with the compatibility relations (28) and the constitutive equations (29) if and only if

$$(\dot{\underline{u}}, \dot{w}, c) \in K \left| \frac{\partial F}{\partial \dot{\underline{u}}}(\tilde{\underline{u}} - \dot{\underline{u}}) + \frac{\partial F}{\partial \dot{w}}(\tilde{w} - \dot{w}) + \frac{\partial F}{\partial c}(\tilde{c} - c) \geq 0, \quad \forall (\tilde{\underline{u}}, \tilde{w}, \tilde{c}) \in K, \right. \quad (33)$$

where  $K$  is the convex set of admissible fields with given boundary conditions and evolution criterion on delamination front and is defined by:

$$K = \left\{ (\dot{\underline{u}}, \dot{w}, c) \mid \begin{array}{ll} \dot{\underline{u}}, \dot{w}, \nabla \dot{w} \mathcal{C}^0(\Omega_t^i), \mathcal{C}_m^1(\Omega_t^i) & i \in \{1, 2\} \\ \dot{\underline{u}}_\beta = \dot{\lambda} u_\beta^d & \text{on } \partial\Omega_{u_\beta}, \beta \in \{1, 2\} \\ \dot{w} = \dot{\lambda} w^d & \text{on } \partial\Omega_w \\ \nabla \dot{w} \cdot \underline{n} = \dot{\lambda} \theta_n^d & \text{on } \partial\Omega_{\theta_n} \\ \llbracket \dot{\underline{u}} \rrbracket^i - h^i \nabla \dot{w}^0 = c \frac{\partial g}{\partial G} (\llbracket \nabla \underline{u} \rrbracket^i - h^i \nabla \nabla w^0) \cdot \underline{y} & i \in \{1, 2\} \\ \llbracket \dot{w} \rrbracket^i = 0 & \text{on } \Gamma_t \\ \llbracket \nabla \dot{w} \rrbracket^i = c \frac{\partial g}{\partial G} \llbracket \kappa \rrbracket^i \cdot \underline{y} & i \in \{1, 2\} \\ c \geq 0 & \text{on } \Gamma_t^{\text{rupt}} \\ c = 0 & \text{on } \Gamma_t \setminus \Gamma_t^{\text{rupt}} \end{array} \right\}.$$

**Proof.** In the following,  $\hat{\varepsilon} = \varepsilon(\hat{\underline{u}}) = \varepsilon(\tilde{\underline{u}} - \dot{\underline{u}})$ ,  $\hat{\kappa} = \nabla \nabla \hat{w} = \nabla \nabla (\tilde{w} - \dot{w})$ ,  $\hat{\phi} = \hat{c} \frac{\partial g}{\partial G} = (\tilde{c} - c) \frac{\partial g}{\partial G}$ . Because  $(\dot{\underline{u}}, \dot{w}, c) \in K$ , one gets relations (15) and (27), if  $\phi = c \frac{\partial g}{\partial G}$ .

Consider the inequality

$$\frac{\partial F}{\partial \underline{u}} \underline{u} + \frac{\partial F}{\partial \hat{w}} \hat{w} + \frac{\partial F}{\partial \hat{c}} \hat{c} \geq 0,$$

with  $(\underline{u}, \hat{w}, c) \in K$  and for any  $(\underline{u}, \hat{w}, \hat{c})$ , admissible with vanishing displacement on the external boundary (where displacement are prescribed) then, by virtue of the notations due to the compatibility relations (28) and the constitutive equations (29),

$$0 \leq \sum_{i=0}^2 \int_{\Omega'_i} [\dot{\mathbf{N}} : \hat{\mathbf{e}} + \dot{\mathbf{M}} : \hat{\mathbf{k}}] da - \lambda \mathcal{P}_{\text{ext}}(\underline{u}, \hat{w}) - \int_{\Gamma_t} \{ \nabla \underline{u}_0 : [\mathbf{N}] + \hat{\mathbf{k}}_0 : ([\mathbf{M}] + h_i \mathbf{N}_i) \} \phi ds \\ - \int_{\Gamma_t} \hat{G}(\underline{u}, \hat{w}) \hat{\phi} ds - \int_{\Gamma_t} G^* \phi \hat{\phi} ds.$$

Using the divergence theorem and the properties of admissible fields  $(\underline{u}, \hat{w})$  on the external boundary,

$$0 \leq \sum_{i=0}^2 \int_{\Omega'_i} [-\text{div} \dot{\mathbf{N}} + \text{div} \text{div} \dot{\mathbf{M}} \underline{e}_3] \cdot (\underline{u} + \hat{w} \underline{e}_3) da + \int_{\partial \Omega_{F_\beta}} [\underline{e}_\beta \cdot \dot{\mathbf{N}} \cdot \underline{n} - \lambda F_\beta^d] \hat{v}_\beta ds \\ + \int_{\partial \Omega_{F_3}} \left[ -\text{div} \dot{\mathbf{M}} \cdot \underline{n} - \frac{\partial \underline{n} \cdot \dot{\mathbf{M}} \cdot \underline{t}}{\partial s} - \lambda F_3^d \right] \hat{w} ds + \int_{\partial \Omega_C} [\underline{n} \cdot \dot{\mathbf{M}} \cdot \underline{n} - \lambda C^d] (\underline{n} \cdot \nabla \hat{w}) ds \\ - \int_{\Gamma} \{ \underline{y} \cdot [\dot{\mathbf{N}} \cdot \underline{u}] + \underline{y} \cdot [\dot{\mathbf{M}} \cdot \nabla \hat{w}] - \underline{y} \cdot [\text{div} \dot{\mathbf{M}}] \hat{w} + \nabla \underline{u}_0 : [\mathbf{N}] \phi + \hat{\mathbf{k}}_0 : ([\mathbf{M}] + h_i \mathbf{N}_i) \phi \} ds.$$

At this step, this inequality must be satisfied for any admissible field  $(\underline{u}, \hat{w}, \hat{c})$ . Consequently, the first four terms, in the previous inequality imply that the rate solution must satisfy Eqs. (30) and (31). The inequality is thus reduced to:

$$0 \leq - \int_{\Gamma_t} \left\{ \underline{y} \cdot [\dot{\mathbf{N}}] \cdot \underline{u}_0 + \underline{y} \cdot [\dot{\mathbf{M}}] \cdot \nabla \hat{w}_0 - \underline{y} \cdot [\text{div} \dot{\mathbf{M}}] \hat{w}_0 \right\} ds \\ - \int_{\Gamma_t} \left\{ \underline{\tau} \cdot \nabla \underline{u}_0 \cdot [\mathbf{N}] \cdot \underline{\tau} \phi + \hat{\mathbf{k}}_0 : ([\mathbf{M}] + h_i \mathbf{N}_i) \phi \right\} ds \\ - \int_{\Gamma_t} \left\{ \underline{y} \cdot \dot{\mathbf{N}} \cdot [\underline{u}]^i + \underline{y} \cdot \dot{\mathbf{M}}_i \cdot [\nabla \hat{w}]^i + \hat{G}(\underline{u}, \hat{w}) \hat{\phi} + G^* \phi \hat{\phi} \right\} ds.$$

Finally, using the compatibility of the velocities  $(\underline{u}, \hat{w})$  with the front velocity  $\hat{\phi} = \hat{c} \frac{\partial \underline{y}}{\partial \hat{G}}$  on  $\Gamma_t$  (because  $(\underline{u}, \hat{w}, c)$  and  $(\underline{u}, \hat{w}, \hat{c})$  are elements of  $K$ ), one obtains

$$0 \leq - \int_{\Gamma} \left\{ \underline{y} \cdot [\dot{\mathbf{N}}] + \frac{\partial}{\partial s} (\phi [\mathbf{N}] \cdot \underline{\tau}) \right\} \cdot \underline{u}_0 ds \\ + \int_{\Gamma} \left\{ \underline{y} \cdot [\dot{\mathbf{M}}] \cdot \underline{y} + h_i \underline{y} \cdot \dot{\mathbf{N}}_i \cdot \underline{y} - \phi \underline{y} \cdot ([\nabla \mathbf{M}] + h_i \nabla \mathbf{N}_i) \cdot \underline{y} \right\} [\nabla \hat{w}_0 \cdot \underline{y}] ds \\ + \int_{\Gamma} \left\{ \frac{\partial \phi}{\partial s} [([\mathbf{M}] + h_i \mathbf{N}_i) : (\underline{\tau} \otimes \underline{y} + \underline{y} \otimes \underline{\tau})] \right\} [\nabla \hat{w}_0 \cdot \underline{y}] ds \\ - \int_{\Gamma} \left( [\text{div} \dot{\mathbf{M}}] \cdot \underline{y} + \frac{\partial}{\partial s} \left\{ \underline{y} \cdot ([\dot{\mathbf{M}}] - h_i \underline{y} \cdot \dot{\mathbf{N}}_i) \cdot \underline{\tau} \right\} + \frac{\partial}{\partial s} \left\{ \underline{\tau} \cdot \frac{\partial}{\partial s} [\phi ([\mathbf{M}] + h_i \mathbf{N}_i)] \cdot \underline{\tau} \right\} \right) \hat{w}_0 ds \\ - \int_{\Gamma} D_\phi(G) \hat{\phi} ds.$$

This inequality must hold for any admissible  $(\underline{u}, \hat{w}, \hat{c})$ , which implies that local equilibrium equations in rate form on  $\Gamma_t$  (16) are fulfilled as well the evolution law (25).

To summarize, the variational inequality (33) has been shown to imply the rate boundary problem ((15), (16), (25), (27), (30) and (31)) assuming the compatibility relations (28) and the constitutive equations (29).

The converse implication is straightforward and Theorem 1 follows.  $\square$

*Rewriting of the rate problem:* The set  $K$  of admissible fields with the kinematic boundary conditions, the Hadamard jump relations (15) and the evolution law of  $\Gamma_t$  is convex.

In the following paragraphs, for given  $\dot{\lambda}$ ,  $\phi$  (i.e. for given  $c$ ), if  $(\dot{u}, \dot{w})$  is solution to Eqs. (15), (16), (27), (30), (31), with (28) and (29) then, let

$$H(c, \dot{\lambda}) = F(\dot{u}(c, \dot{\lambda}), \dot{w}(c, \dot{\lambda}), c, \dot{\lambda})$$

denote the value of  $F$  for the solution of the rate problem, when the front velocity and the loading evolution are prescribed via  $c$  and  $\dot{\lambda}$ . The evolution of the delamination front is governed by

$$\frac{\partial H}{\partial c}[c, \dot{\lambda}](\tilde{c} - c) \geq 0, \quad \forall \tilde{c} \text{ admissible with evolution law (23) and criterion (25).}$$

In the sequel, we denote by  $\tilde{H}$  the second-order derivative of functional  $H$  with respect to  $c$ . Because the functional  $H$  is a quadratic function of  $(c, \dot{\lambda})$ ,

$$\frac{\partial H}{\partial c}[c, \dot{\lambda}] = \frac{\partial^2 H}{\partial c^2} c + \frac{\partial^2 H}{\partial c \partial \dot{\lambda}} \dot{\lambda}$$

and the evolution inequality can be written as:

find  $c \in V$  admissible with (23) and (25) such as

$$(\tilde{c} - c) \left\{ \tilde{H}c + \frac{\partial^2 H}{\partial c \partial \dot{\lambda}} \dot{\lambda} \right\} \geq 0, \quad \forall \tilde{c} \text{ admissible with (23) and (25).} \quad (34)$$

The functional  $\tilde{H}$  is defined on the current configuration. It is a function of the displacement and of the position of delamination front at time  $t$ . So, the instantaneous evolution of the structure from its configuration at time  $t$  depends solely on the current geometry and the state of equilibrium at time  $t$ .

#### 4. Characterization of the current state

In this section, we investigate whether there exists one or more admissible solutions to the quasi-static instantaneous evolution, when the loading parameter increases. To achieve this goal, general results from Duvaut and Lions (1976) and the same framework as in Nguyen (1993) are used.

In what follows, “stability” refers to the existence of a solution to the rate problem and “non-bifurcation” to the uniqueness of the solution of the same problem. More precisely, “non-bifurcation” means that the solution to the system of local equations (15), (16) and (25)–(31) is unique. This definition of non-bifurcation thus does not consider bifurcations of higher order in time.

##### 4.1. Stability of the evolution

**Theorem 2.** *The evolution of the delaminated structure from a state, characterized by a displacement  $(\underline{u}, w)$  and a delamination front  $\Gamma_t$ , is stable (has at least one solution) if and only if*

$$c\tilde{H}c > 0, \quad \forall c \in V - \{0\},$$

where  $V$  is the set of admissible fields with evolution laws, defined by (26).

**Proof.** The evolution of the system is associated to the solution  $c \in V$  of Eq. (34). Taking into account different admissible fields  $c'$ , it leads to:

$$c \in V \quad \text{and} \quad \tilde{H}c + \frac{\partial^2 H}{\partial c \partial \lambda} \dot{\lambda} = 0. \quad (35)$$

Then, to solve (35) for a prescribed load increase  $\dot{\lambda}$ , one has to invert  $\tilde{H}$ . If  $\tilde{H}$  is positive definite on  $V$ , then (35) can be solved for any given  $\dot{\lambda}$ . Obviously, if one is able to find  $c$  associated to any  $\dot{\lambda}$  by Eq. (35), then the function  $c \rightarrow \tilde{H}c$  is surjective on  $V$ .

Moreover, let the dead load be defined as the first value of the loading parameter that enables delamination propagation without a loading parameter evolution i.e. such that there exists  $c \in V - \{0\}$  such as  $\tilde{H}c = 0$ . Consequently, the dead load indicates the possibility of an unstable evolution where a delamination propagation may occur without a load increase.  $\square$

#### 4.2. Non-bifurcation of evolution

**Theorem 3.** *There is no bifurcation (at most one solution) in the evolution of the delaminated structure if and only if*

$$c\tilde{H}c > 0 \quad \forall c \in \text{span}(V) - \{0\},$$

where  $\text{span}(V)$  is the vector space generated by  $V$ .

**Proof.** Consider two different solutions  $c_1$  and  $c_2$  to (34). Then, we get:

$$(c_1 - c_2) \frac{\partial^2 H}{\partial c \partial \lambda} \dot{\lambda} \geq (c_2 - c_1) \tilde{H}c_2 \quad \text{and} \quad (c_2 - c_1) \frac{\partial^2 H}{\partial c \partial \lambda} \dot{\lambda} \geq (c_1 - c_2) \tilde{H}c_1. \quad (36)$$

Consequently, it gives:  $(c_1 - c_2)\tilde{H}(c_1 - c_2) = 0$ , where  $(c_1 - c_2)$  belongs to  $\text{span}(V)$ . If  $\tilde{H}$  is definite positive on  $\text{span}(V)$ , then the previous equation implies  $c_1 = c_2$ , which contradicts the hypothesis of two distinct solutions.

Otherwise,  $(c_1 - c_2)$  belongs to  $\text{Ker}(\tilde{H})$ , subset of  $\text{span}(V)$  where  $\tilde{H}$  is not definite.  $\square$

### 5. Application to von Karman plate theory

Application to von Karman plates (with large transverse displacement and large deformation) (Love, 1944; Timoshenko and Woinowsky-Krieger, 1959) is important because it allows coupling between delamination and buckling. The approach used in Sections 3 and 4 for Kirchhoff–Love plates can be extended to von Karman plates in a straightforward way. The difference between the Kirchhoff–Love and von Karman theories is mainly due to the non-linearity of the plane strain with respect to the transverse displacement in von Karman plates:

$$\boldsymbol{\varepsilon}(\underline{u}, w) = \frac{1}{2}(\nabla \underline{u} + {}^T \nabla \underline{u} + \nabla w \otimes \nabla w), \quad \boldsymbol{\kappa}(\underline{u}) = \nabla \nabla w. \quad (37)$$

This enables to take buckling into account in von Karman theory.

Upon following the same steps as in Sections 3 and 4, one obtains the corresponding evolution formulation for von Karman plates, whose most important aspects are summarized in this section.

### 5.1. Study of the equilibrium state

#### 5.1.1. Equilibrium equations

Stationarity of the Lagrangian with kinematic constraints described by (2) on  $\Gamma_t$  gives, in addition to the kinematic relations on  $\partial\Omega$  (12), the compatibility relations in  $\Omega_t$  (37) and the constitutive relation (3), the following equilibrium equations:

- in  $\Omega_t^i$  with  $i \in \{0, 1, 2\}$ ,

$$\begin{aligned} \operatorname{div} \mathbf{N} &= \underline{0}, \\ \operatorname{div} \operatorname{div} \mathbf{M} - \operatorname{div}(\mathbf{N} \cdot \nabla w) &= 0, \end{aligned} \quad (38)$$

- along  $\Gamma_t$ ,

$$\begin{aligned} [\mathbf{N}] \cdot \underline{v} &= \underline{0}, \\ [\operatorname{div} \mathbf{M}] \cdot \underline{v} + \frac{\partial}{\partial s} [\underline{v} \cdot \{[\mathbf{M}] + h^i \mathbf{N}^i\} \cdot \underline{t}] &= 0, \\ \underline{v} \cdot ([\mathbf{M}] + h^i \mathbf{N}^i) \cdot \underline{v} &= 0, \end{aligned} \quad (39)$$

- on the external boundary,

$$\begin{aligned} \underline{\varepsilon}_\beta \cdot \mathbf{N} \cdot \underline{n} &= \lambda F_\beta^d \quad \text{on } \partial\Omega_{F_\beta}, \quad \forall \beta \in \{1, 2\}, \\ -\operatorname{div} \mathbf{M} \cdot \underline{n} - \frac{\partial}{\partial s} (\underline{n} \cdot \mathbf{M} \cdot \underline{t}) &= \lambda F_3^d \quad \text{on } \partial\Omega_{F_3}, \\ \underline{n} \cdot \mathbf{M} \cdot \underline{n} &= \lambda C^d \quad \text{on } \partial\Omega_C. \end{aligned} \quad (40)$$

#### 5.1.2. Energy release rate

Analyzing the total dissipation of the system, we get:

$$G = [\mathbf{W}] - \underline{v} \cdot (\mathbf{N}^i \cdot [\nabla \underline{u}]^i + \mathbf{M}^i \cdot [\nabla \nabla w]^i - h^i \mathbf{N}^i \cdot \nabla \nabla w_0) \cdot \underline{v}. \quad (41)$$

### 5.2. Evolution of delamination front

#### 5.2.1. Rate boundary value problem

The evolution problem for the instantaneous delamination induced by a load increment  $\dot{\lambda}$  is defined again by (15), (16), (23), (25), (27), (28), (29), (30), (31), except for the following differences:

- in plate  $i$ , (28) is changed in

$$\dot{\mathbf{e}}^i = \frac{1}{2} (\nabla \underline{u}^i + {}^T \nabla \underline{u}^i + \nabla w^i \otimes \nabla \dot{w}^i + \nabla \dot{w}^i \otimes \nabla w^i), \quad \dot{\mathbf{k}}^i = \nabla \nabla \dot{w}^i, \quad (42)$$

- in plate  $i$ , (30) is replaced by

$$\begin{aligned} \operatorname{div} \dot{\mathbf{N}} &= \underline{0}, \\ \operatorname{div} \operatorname{div} \dot{\mathbf{M}} - \operatorname{div}(\dot{\mathbf{N}} \cdot \nabla w + \mathbf{N} \cdot \nabla \dot{w}) &= 0, \end{aligned} \quad (43)$$

- on  $\Gamma_t$ , the equilibrium conditions in rate form (16) along the moving delamination front  $\Gamma_t$  become

$$\begin{aligned}
\llbracket \dot{\mathbf{N}} \rrbracket \cdot \underline{y} &= -\frac{\partial}{\partial s} \{ \phi \llbracket \mathbf{N} \rrbracket \cdot \underline{\tau} \}, \\
\llbracket \operatorname{div} \dot{\mathbf{M}} \rrbracket \cdot \underline{y} + \frac{\partial}{\partial s} \{ \underline{y} \cdot (\llbracket \dot{\mathbf{M}} \rrbracket + h^i \dot{\mathbf{N}}^i) \cdot \underline{\tau} \} &= -\frac{\partial}{\partial s} \left\{ \underline{\tau} \cdot \frac{\partial}{\partial s} (\phi \llbracket \mathbf{M} \rrbracket + h_i \mathbf{N}_i) \cdot \underline{\tau} \right\} + \phi \llbracket \mathbf{N} : \nabla \nabla w \rrbracket, \quad (44) \\
\underline{y} \cdot (\llbracket \dot{\mathbf{M}} \rrbracket + h^i \dot{\mathbf{N}}^i) \cdot \underline{y} &= \phi \llbracket \underline{y} \cdot \nabla (\llbracket \mathbf{M} \rrbracket + h^i \mathbf{N}^i) \cdot \underline{y} \rrbracket - \{ \llbracket \mathbf{M} \rrbracket + h^i \mathbf{N}^i \} : (\underline{\tau} \otimes \underline{y} + \underline{y} \otimes \underline{\tau}) \frac{\partial \phi}{\partial s}.
\end{aligned}$$

During the propagation of the delamination front, the criterion  $g(G) = 0$  has to be maintained. So,  $\frac{\partial g}{\partial G} D_\phi(G) = 0$  where

$$D_\phi(G) = \tilde{G}(\dot{\mathbf{N}}, \dot{\mathbf{M}}) + \hat{G}(\dot{\underline{u}}, \dot{w}) + \phi G^*$$

with now

$$\tilde{G}(\dot{\mathbf{N}}, \dot{\mathbf{M}}) = -\underline{y} \cdot \dot{\mathbf{N}} \cdot (\llbracket \nabla \underline{y} \rrbracket^i - h^i \nabla \nabla w_0) \cdot \underline{y} - \underline{y} \cdot \dot{\mathbf{M}} \cdot \llbracket \nabla \nabla w \rrbracket^i \cdot \underline{y},$$

$$\hat{G}(\dot{\underline{u}}, \dot{w}) = \llbracket \mathbf{N} \rrbracket : (\nabla \dot{\underline{u}}_0 + \nabla w_0 \otimes \nabla \dot{w}_0) + (\llbracket \mathbf{M} \rrbracket + h^i \mathbf{N}^i) : \nabla \nabla \dot{w}_0,$$

$$\begin{aligned}
G^* &= -\llbracket \mathbf{N} : (\nabla \nabla \underline{y} + \nabla w \otimes \nabla \nabla w) + \mathbf{M} : \nabla \nabla \nabla w \rrbracket \cdot \underline{y} + \underline{y} \cdot \nabla \mathbf{N}^i \cdot \underline{y} \cdot (\llbracket \underline{y} \rrbracket^i - h^i \nabla \nabla w_0) \cdot \underline{y} \\
&\quad + \underline{y} \cdot \nabla \mathbf{M}^i \cdot \underline{y} \cdot \llbracket \nabla \nabla w \rrbracket^i \cdot \underline{y} + \mathbf{N}^i : (\llbracket \nabla \nabla \underline{y} \rrbracket^i - h^i \nabla \nabla \nabla w_0) + \mathbf{M}^i : \llbracket \nabla \nabla \nabla w \rrbracket^i.
\end{aligned}$$

### 5.2.2. Global formulation

As in the Kirchhoff–Love case, the local equations of the rate problem are equivalent to a variational inequality of the form (33), with the functional  $F$  now given by:

$$F(\dot{\underline{u}}, \dot{w}, c, \lambda) = \sum_{i=0}^2 \int_{\Omega'_i} \frac{1}{2} [\dot{\boldsymbol{\varepsilon}}, \dot{\boldsymbol{\kappa}}] W''' \begin{bmatrix} \dot{\boldsymbol{\varepsilon}} \\ \dot{\boldsymbol{\kappa}} \end{bmatrix} d\omega - \lambda \mathcal{P}_{\text{ext}}(\dot{\underline{u}}, \dot{w}) - \int_{\Gamma_t} \hat{G}(\dot{\underline{u}}, \dot{w}) c \frac{\partial g}{\partial G} ds + \frac{1}{2} \int_{\Gamma_t} G^* c^2 \left( \frac{\partial g}{\partial G} \right)^2 ds,$$

where  $\dot{\boldsymbol{\varepsilon}}$  and  $\dot{\boldsymbol{\kappa}}$  are given by (42).

## 6. Applications

Two analytical applications are presented in this section.

The first one (Section 6.1) corresponds to the bending of a delaminated plate. This structure is an extension of the double cantilever beam considered in Roudolff and Ousset (2002), La Saponara et al. (2002), Greco et al. (2002). Here, analytical results concerning the energy release rate are given as well as some conclusions on the stability of the evolution.

The second example (Section 6.2) is a delaminated structure under compression. It uses the non-linear von Karman plate theory and we study the propagation of delamination on a frame of plates after buckling of the plates below and above the fracture. This phenomenon induces coupling between delamination and buckling as shown in Moon et al. (2002), Nilsson et al. (2001) and Storakers and Nilsson (1993). In what follows, we first obtain an analytical form of the energy release rate with or without buckling. Then, due to the global functional introduced to this end in the previous section, the linear propagation of the delamination front is discussed in terms of stability and bifurcation.

## 6.1. Propagation of delamination in a double cantilever beam

### 6.1.1. Definition of the structure

The structure considered is a rectangular plate (thickness  $2e$ ) with a fracture in its median plane. The constitutive material is assumed to be homogeneous, isotropic and elastic with characteristics denoted by  $E$ , the Young modulus,  $\nu$ , the Poisson ratio.

The rectangular plate  $\Omega_t$  is defined by:

$$\Omega_t = \left\{ (x_1, x_2) \mid x_1 \in [0, L], x_2 \in \left[ -\frac{b}{2}, \frac{b}{2} \right] \right\}.$$

It is delaminated throughout the whole axis  $\underline{e}_2$ . The delamination front is located on the line defined by  $x_1 = a$ ,  $x_2 \in [-\frac{b}{2}, \frac{b}{2}]$  ( $a < L$ ).  $a_{\text{ini}}$  denotes the initial location of the delamination front.

Let  $\Omega_t^1$  (respectively  $\Omega_t^2$ ) denote the plate above (resp. below) the delamination and  $\Omega_t^0$  the undamaged plate.

The loading of the structure is the following (see Fig. 2):

- the edge  $x_1 = L$  is clamped (displacements prescribed to zero),
- on  $x_2 = \pm \frac{b}{2}$ , the normal and shear stresses vanish and the rotation is given equal to 0, i.e.

$$N \cdot \underline{e}_2 = 0, \quad \text{div } \mathbf{M} \cdot \underline{e}_2 + (\underline{e}_1 \cdot \mathbf{M} \cdot \underline{e}_2)_{,1} = 0, \quad w_{,2} = 0,$$

- on the remaining boundary ( $x_1 = 0$ ), if  $2w_d$  denotes the crack opening displacement for the delaminated structure,

$$F_1^d = F_2^d = 0, \quad C^d = 0, \quad w^2 = -w_d \quad \text{and} \quad w^1 = w_d.$$

### 6.1.2. Equilibrium

The displacement vector and the stress tensor are easily found to be independent on the  $x_2$  coordinate. The configuration is symmetrical with respect to the  $x_1$  axis. So, we get for the solution:

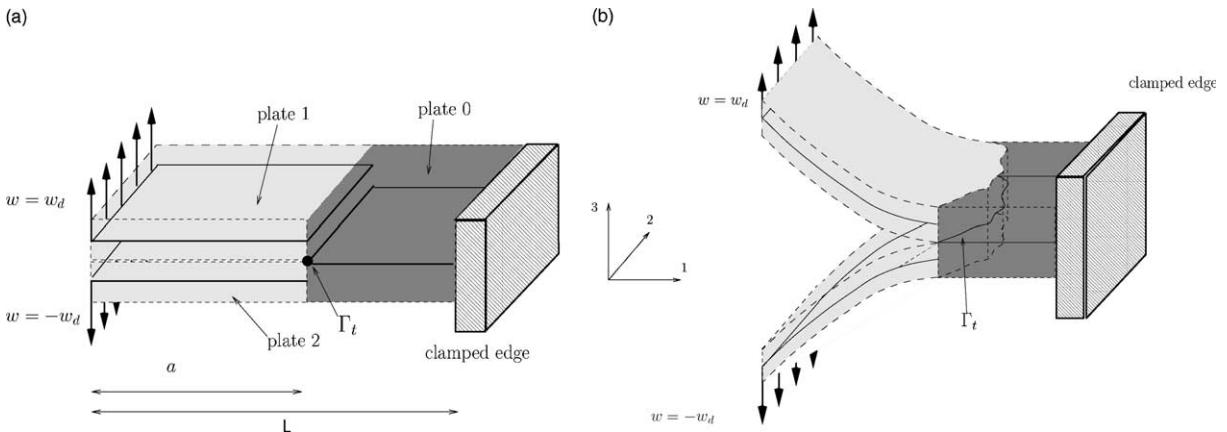


Fig. 2. Double cantilever beam and loading: (a) initial and undeformed configuration, (b) deformation of the structure under loading with propagation of the delamination front.

$$\begin{aligned}\underline{u}^1(x_1, x_2) &= \underline{u}^2(x_1, x_2) = \underline{0}, \\ w^1(x_1, x_2) &= w^2(x_1, x_2) = w_d \left[ 1 - \frac{3x_1}{2a} + \frac{x_1^3}{2a^3} \right], \\ \underline{u}^0(x_1, x_2) &= \underline{0}, \\ w^0(x_1, x_2) &= 0\end{aligned}$$

and

$$\mathbf{N}^0 = \mathbf{N}^1 = \mathbf{N}^2 = \mathbf{0},$$

$$\mathbf{M}^0 = \mathbf{0},$$

$$\mathbf{M}^1 = -\mathbf{M}^2 = \frac{3\mathcal{D}w_d x_1}{a^3} (\underline{e}_1 \otimes \underline{e}_1 + v \underline{e}_2 \otimes \underline{e}_2) \quad \text{with } \mathcal{D} = \frac{Ee^3}{12(1-v^2)}.$$

The potential energy at a state of equilibrium is  $P(w_d, a) = \frac{3b\mathcal{D}w_d^2}{a^3}$ .

### 6.1.3. Energy release rate and linear propagation

The energy release rate associated to a delamination whose current length is denoted by  $a$  is:

$$G = \frac{9\mathcal{D}w_d^2}{a^4}.$$

If a Griffith-like criterion is adopted, such as  $\phi(G - G_C) = 0$  with  $\phi \geq 0$  and  $G \leq G_C$  on  $\Gamma_t$ , the critical loading is reached for  $w_d = w_C(a) = \sqrt{\frac{G_C a^4}{9\mathcal{D}}}$ .

Considering only uniform propagations of delamination, which means that the normal velocity of propagation is constant along  $\Gamma_t$  ( $\phi(s) = \phi(x_2) = \dot{a}$ ), the analysis of evolution consists in studying  $G$  as a function of  $a$ . Then, the normal velocity of delamination front is given by  $D_\phi(G) = 0$ , which is equivalent to  $\frac{\partial G}{\partial w_d} \dot{w}_d + \frac{\partial G}{\partial a} \dot{a} = 0$ . This leads to  $\frac{\dot{a}}{a} = \frac{1}{2} \frac{\dot{w}_d}{w_d}$ .

Consequently, the evolution with uniform propagation is always stable, when the loading is driven by displacement, as shown on Figs. 3, 4 and 5.

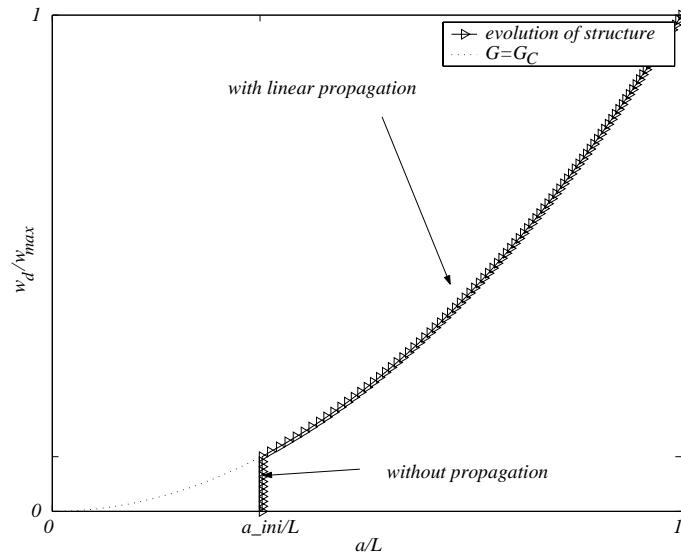


Fig. 3. Evolution of the delamination length with loading in DCB experiment ( $a_{ini}$  denotes initial length of the delamination).

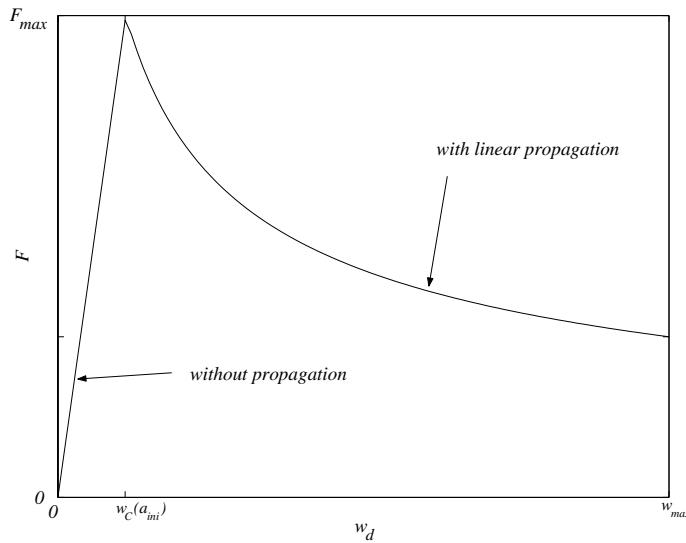


Fig. 4. Load/deflection curve with linearly propagating delamination (DCB experiment).

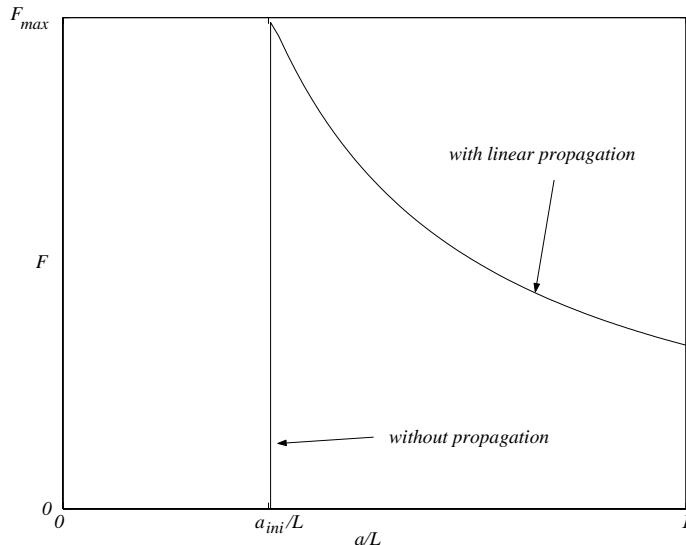


Fig. 5. Resistance curves: transverse force versus delamination length (case of linear propagation) (DCB experiment).

This result could have been established from the global functional  $H$  which here takes the following form:

$$H(\dot{w}_d, \dot{a}) = \frac{3b\mathcal{D}(\dot{w}_d)^2}{a^3} + 18b\mathcal{D}\frac{(\dot{w}_d)^2}{a^3} \left[ \frac{\dot{a}}{a} - \frac{\dot{w}_d}{w_d} \right] \frac{\dot{a}}{a}.$$

#### 6.1.4. Analysis of evolution and general propagation

Let now consider general evolutions of the delamination front when the propagation criterion is reached. Any normal velocity is then sought in the form:

$$\phi(s) = \phi(x_2) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(2n\pi \frac{x_2}{b}\right) + \sum_{n=1}^{\infty} b_n \sin\left(2n\pi \frac{x_2}{b}\right)$$

and the uniqueness of the  $(a_n, b_n)$  coefficients is investigated.

Analysis of the rate boundary value problem ((15), (16), (27)–(31)) implies that the propagation velocity necessarily takes the form  $\phi(s) = \phi(x_2) = \sum_{n=0}^{\infty} a_n \cos(2n\pi \frac{x_2}{b})$ .

Then, we get the following displacement rate:

$$\begin{aligned} \dot{\underline{u}}^1 &= \dot{\underline{u}}^2 = \underline{0} \quad \text{and} \quad \dot{\underline{u}}^0 = \underline{0}, \\ \dot{w}^0 &= 0, \\ \dot{w}^1 &= -\dot{w}^2 = \dot{w}_d \left[ 1 - \frac{3x_1}{2a} + \frac{x_1^3}{2a^3} \right] + \frac{3a_0 w_d}{2a} \left[ \frac{x_1}{a} - \frac{x_1^2}{a^2} \right] \\ &\quad + \frac{3w_d}{a^2} \sum_{n=1}^{\infty} a_n \frac{\left[ a \sinh\left(\frac{2n\pi x_1}{b}\right) \cosh\left(\frac{2n\pi a}{b}\right) - x_1 \cosh\left(\frac{2n\pi x_1}{b}\right) \sinh\left(\frac{2n\pi a}{b}\right) \right]}{\cosh\left(\frac{2n\pi a}{b}\right) \sinh\left(\frac{2n\pi a}{b}\right) - 2n\pi \frac{a}{b}} \cos\left(\frac{2n\pi x_2}{b}\right). \end{aligned}$$

As introduced in Section 4, stability and non-bifurcation are characterized by positive definiteness of the functional  $H$  over the set  $V$  of admissible delamination front normal velocities or over the vector space  $\text{span}(V)$ , respectively.

Here,  $H$  corresponds to:

$$\begin{aligned} H(\phi, \dot{w}_d) &= H(a_0, a_1, \dots, a_n, \dots, \dot{w}_d) \\ &= \frac{3b\mathcal{D}\dot{w}_d^2}{a^3} - \frac{18b\mathcal{D}w_d\dot{w}_d}{a^4} + 18 \frac{b\mathcal{D}w_d^2 a_0^2}{a^5} + \frac{9b\mathcal{D}w_d^2}{a^5} \sum_{n=1}^{\infty} a_n^2 \left[ 1 + 2 \frac{\frac{2n\pi a}{b} \sinh^2\left(\frac{2n\pi a}{b}\right)}{\sinh\left(\frac{2n\pi a}{b}\right) \cosh\left(\frac{2n\pi a}{b}\right) - \frac{2n\pi a}{b}} \right]. \end{aligned}$$

Consequently, the second derivative of  $H$  with respect to  $\phi$  is positive definite for all aspect ratios  $a/b$  and there is no bifurcation in the evolution of the delamination front.

#### 6.1.5. Discussion

The foregoing analysis of the Double Cantilever Beam gives results similar to experimental observations in La Saponara et al. (2002) and Roudolff and Ousset (2002). One should note that this modeling cannot deal with either the start of delamination ( $a \ll L$ ) or the case  $a \approx L$ , i.e. when the delamination front reaches the clamped edge, because edge effects are not taken into account.

### 6.2. Delaminated plate under compression

#### 6.2.1. Definition of the structure

A rectangular plate

$$\Omega_t = \left\{ (x_1, x_2) \text{ with } x_1 \in [-L, L], x_2 \in \left[ -\frac{b}{2}, \frac{b}{2} \right] \right\},$$

assumed to be homogeneous, isotropic and elastic ( $E, v$ ), is considered in this example.

It is delaminated throughout the whole axis  $\underline{x}_2$ . The delamination is in the middle of the plate. Its front is initially located on the lines defined by

$$x_1 = a_1, \quad x_2 \in \left[ -\frac{b}{2}, \frac{b}{2} \right]$$

and

$$x_1 = a_2, \quad x_2 \in \left[ -\frac{b}{2}, \frac{b}{2} \right] \quad (|a_i| < L \text{ and } a_1 > a_2).$$

Let  $2l$  denote to the length of the delaminated part (i.e.  $a_1 - a_2 = 2l$ ) (see Fig. 6).

The loading of the structure is as follows (see Fig. 6):

- on the transverse boundary (i.e.  $x_2 = \pm \frac{b}{2}$ ),  $\mathbf{N} \cdot \underline{e}_2 = \underline{0}$ ,  $\operatorname{div} \mathbf{M} \cdot \underline{e}_2 + (\underline{e}_1 \cdot \mathbf{M} \underline{e}_2)_{,1} = 0$ ,  $w_{,2} = 0$ ,
- along the edges  $x_1 = \pm L$  the boundary conditions are:
 
$$\begin{aligned} \underline{u} \cdot \underline{e}_1 &= -u_d, \quad F_2^d = F_3^d = 0, \quad C^d = 0 \quad \text{on } x_1 = L, \\ \underline{u} \cdot \underline{e}_1 &= u_d, \quad F_2^d = F_3^d = 0, \quad C^d = 0 \quad \text{on } x_1 = -L \end{aligned} \quad \text{for } x_2 \in \left[ -\frac{b}{2}, \frac{b}{2} \right], \quad (45)$$
- there is no surface force.

In order to take buckling into account, the von Karman plate theory is used and the evolution problem is therefore formulated based on the results of Section 5.

Let  $\mathcal{C}$  and  $\mathcal{D}$  denote the classical ratios  $\frac{Ee}{(1-v^2)}$  and  $\frac{Ee^3}{12(1-v^2)}$ , where  $e$  is the thickness of the plates above and below the delamination.

### 6.2.2. Equilibrium

Using the local equations given in paragraph 5.1.1, two possibilities arise:

- If  $u_d \leq u_B$  with  $u_B = \frac{L\pi^2\mathcal{D}}{P\mathcal{C}(1-v^2)} = \frac{Le^2\pi^2}{12l^2(1-v^2)}$ , then the displacement and the stresses are:
 
$$\underline{u}(x_1, x_2) = \frac{u_d}{L}(x_1\underline{e}_1 - vx_2\underline{e}_2), \quad w = 0,$$

$$\mathbf{N} = Ee \frac{u_d}{L} \underline{e}_1 \otimes \underline{e}_1, \quad \mathbf{M} = \mathbf{0}.$$

The potential energy at equilibrium is in this case independent of  $l$ :

$$P(u_d, l) = 2\mathcal{C}Lb(1-v^2) \left( \frac{u_d}{L} \right)^2.$$

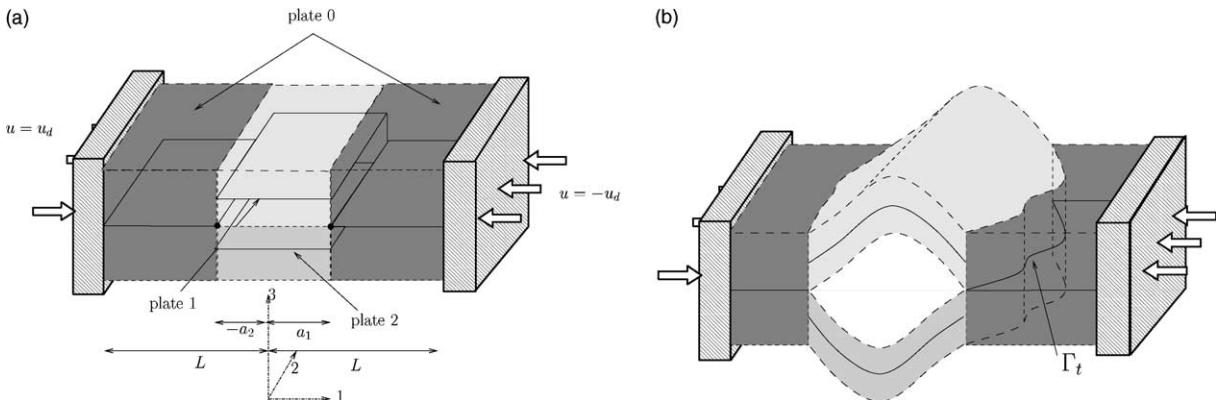


Fig. 6. Delaminated plate under axial compression: (a) initial and undeformed configuration, (b) post-buckling of the loaded structure with propagation of the delamination front.

Consequently, and this is well known (Storakers and Anderson, 1988; Storakers and Nilsson, 1993), no propagation of the delamination can occur without buckling of the structure.

- As soon as  $u_d \geq u_B$ , buckling of the plates above and below delamination occurs. Then, the equilibrium solution becomes:

$$\underline{u}^0 = \left[ u_d - \frac{u_B}{L} (x_1 + L) \right] \underline{e}_1 + v \frac{u_B}{L} x_2 \underline{e}_2 \quad \text{if } x_1 \leq a_2,$$

$$\underline{u}^0 = \left[ -u_d - \frac{u_B}{L} (x_1 - L) \right] \underline{e}_1 + v \frac{u_B}{L} x_2 \underline{e}_2 \quad \text{if } x_1 \geq a_1,$$

$$\underline{u}^1 = \left[ -\frac{u_d}{l} \tilde{x}_1 - \frac{u_B}{L} \left\{ x_1 - \frac{L}{l} \tilde{x}_1 \right\} + \frac{(u_d - u_B)}{2\pi} \sin \left( 2\pi \frac{\tilde{x}_1}{l} \right) \right] \underline{e}_1 + v \frac{u_B}{L} x_2 \underline{e}_2 = \underline{u}^2 \quad \text{if } a_2 \leq x_1 \leq a_1,$$

$$w^0 = 0 \quad \text{if } x_1 \notin [a_2, a_1],$$

$$w^1 = -w^2 = \frac{2l}{\pi} \sqrt{\frac{u_d - u_B}{l}} \left[ \cos \left( \frac{\pi}{l} \tilde{x}_1 \right) + 1 \right] \quad \text{if } a_2 \leq x_1 \leq a_1$$

with  $\tilde{x}_1 = x_1 - \frac{a_1 + a_2}{2}$ .

The generalized stresses are given by:

$$\mathbf{N}^0 = 2\mathbf{N}^1 = 2\mathbf{N}^2 = 2Ee \frac{u_B}{L} \underline{e}_1 \otimes \underline{e}_1 = 2\mathcal{C}(1 - v^2) \frac{u_B}{L} \underline{e}_1 \otimes \underline{e}_1,$$

$$\mathbf{M}^0 = \mathbf{0},$$

$$\mathbf{M}^1 = -\mathbf{M}^2 = \mathcal{D} \frac{2\pi}{l} \sqrt{\frac{u_d - u_B}{l}} \cos \left( \frac{\pi}{l} \tilde{x}_1 \right) [\underline{e}_1 \otimes \underline{e}_1 + v \underline{e}_2 \otimes \underline{e}_2].$$

The potential energy at equilibrium is equal to:

$$P(u_d, l) = 2\mathcal{C}Lb(1 - v^2) \left( \frac{u_B}{L} \right)^2 + \mathcal{D}b \frac{4\pi^2}{l^2} (u_d - u_B).$$

It should be emphasized that only local buckling is considered. Global buckling could occur but is not taken into account here. Influence of this phenomenon on delamination is studied in Qiu et al. (2001), Nilsson et al. (2001) and Hutchinson et al. (2000).

### 6.2.3. Energy release rate and linear propagation

When the loading parameter ( $u_d$ ) is larger than its buckling value  $u_B$ , the energy release rate is:

$$G(a_1) = G(a_2) = \mathcal{D} \frac{4\pi^2(u_d - u_B)}{l^3}.$$

Consequently, the onset of propagation (governed by the Griffith criterion) corresponds to  $u_d = u_C = u_B + G_C \frac{l^3}{4\mathcal{D}\pi^2}$ .

If only uniform propagation is considered (which corresponds to  $\dot{a}_i(s) = \dot{a}_i$ ), then it is sufficient to study the first-order derivative of  $G$  with respect to  $a_i$ .

In this case, the evolution of delamination is stable if  $l \geq l_{\text{stab}}$ , with  $l_{\text{stab}} = \sqrt[5]{\frac{Ee^5 n^4 L}{54G_C(1-v^2)^2}}$ , and unstable otherwise (see Fig. 7). These results are confirmed by a global analysis using the functional  $H$ , here given by:

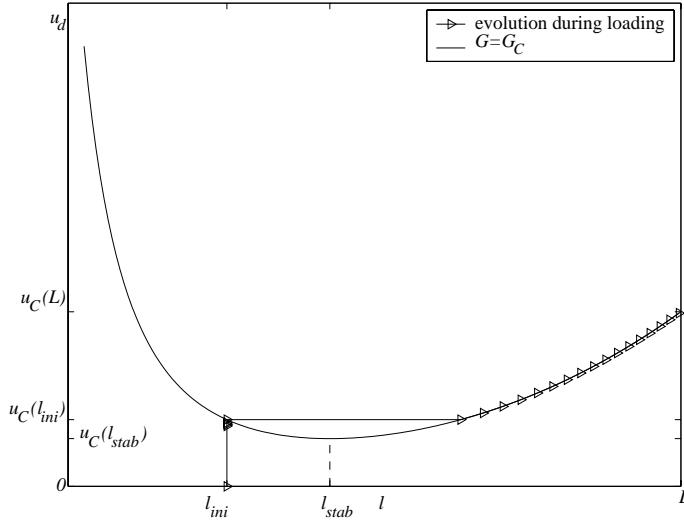


Fig. 7. Loading  $u_d$  versus delamination length  $l$  for the delaminated structure under compression.

$$H(\dot{u}_d, \dot{a}_1, \dot{a}_2) = \frac{\mathcal{D}b\pi^2}{l^3} [-\dot{u}_d(\dot{a}_1 - \dot{a}_2) + (3u_d - 5u_B)(\dot{a}_1 - \dot{a}_2)^2].$$

The fact that  $H$  depends on  $\dot{a}_1 - \dot{a}_2$  clearly indicates that the evolution is not unique because the energy release rate depends only on the total length of delamination whose velocity is  $\dot{l} = \dot{a}_1 - \dot{a}_2$ .

Considering the case of the symmetric uniform propagation where  $a_1 = -a_2$  and  $\dot{a}_1 = -\dot{a}_2$ , the propagation associated to the evolution of the loading parameter gives the structure deformation shown in Fig. 8, where  $l_{\text{ini}}$  represents the initial length of the delamination with  $l_{\text{ini}} < l_{\text{stab}}$ . This result is similar to that of Petitniot and Fabis (1989), for a pre-delaminated plate with an initial penny-shape crack, under

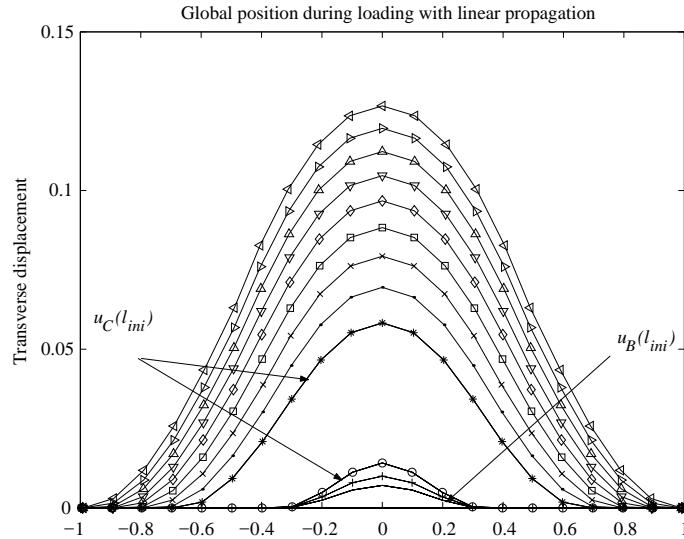


Fig. 8. Evolution of the delaminated structure under compression.

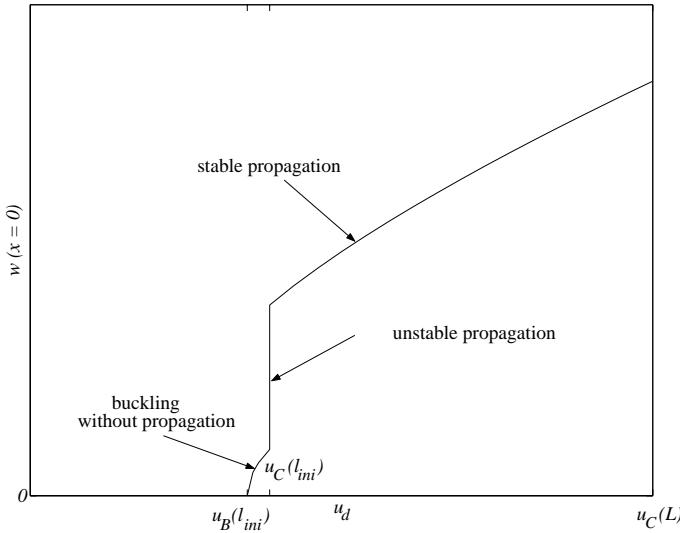


Fig. 9. Out-of-plane displacement in the center of the delamination versus load (displacement  $u_d$ ) for the delaminated structure under compression.

compression: after buckling from  $u_d = u_B(l_{ini})$  to  $u_d = u_C(l_{ini})$ , an unstable propagation occurs at  $u_d = u_C(l_{ini})$ , then a stable propagation takes place until the whole structure is delaminated ( $l = L$ ).

Moreover, the variation of out-of plane displacement with respect to the load, as depicted in Fig. 9, is qualitatively identical to that obtained by Nilsson et al. (2001) from numerical or experimental results.

## 7. Conclusions

The global formulation introduced in this paper enables to study stability and non-bifurcation of the evolution of a delaminated structure under quasi-static loading.

Thanks to the expression on the current configuration, the update configuration is not needed. Moreover, the symmetric form of the global functional  $F$  facilitates the implementation of this approach in numerical schemes such as the FEM. The analysis of the existence and the uniqueness of the solution to the rate boundary value problem would be associated to the study of eigenvalues (positiveness) of the second derivative of the functional  $H$  restricted to admissible fields or to the vector set generated by them. Some elements about the FEM implementation are given in Appendix A.

The analytical examples discussed in this article give good qualitative results compared to experiments even if only very simple models were considered with or without buckling.

Introduction of other kinematic relations can be considered (Anquez et al., 1990). It would introduce different equilibrium relations, and lead to another form of the energy release rate. Consequently, the global functional would have to be modified accordingly as well.

As previously shown with three-dimensional structures with irreversible mechanical transformation governed by energetic criterion, this framework can be generalized to elastic interface (Borg et al., 2002) as was also considered in Pradeilles-Duval (1992).

If a three-dimensional body including planar cracks is considered as in Nguyen et al. (1990) or Bonnet (1999), an extension of this framework can be given if the domains above and below the crack or the undamaged domain can be considered as plates at least in the neighborhood of the crack front. Then, even if a

more general than Kirchhoff–Love or von Karmann plates behavior has to be considered and if the propagation in the plane of the crack is assumed, then the approach used in Sections 3 and 4 can be extended.

Moreover, the present formulation can easily be generalized to plates with inelastic constitutive behavior, together with a free surface energy. In this case, one obtains an additional relation on  $\Gamma_t$  arising due to the possibility of simultaneous evolution of internal variables and propagation of the delamination front as has been discussed in Pradeilles-Duval and Stolz (1995). Consequently, either the front would propagate, or the internal variables rates would change in each plate on both sides.

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## Appendix A. Numerical implementation

In this section, a numerical solution procedure is presented. It includes remarks on the delamination front representation and the interpolation of the normal velocity on the front. Discretized equations are given for the rate boundary value problem. After comments on the stability and non-bifurcation criteria, the computation of the actual extension velocity is considered.

### A.1. Representation of the delamination front and its normal velocity

Implementations have already been proposed in Nguyen et al. (1990) (based on 2-D FEM with plane strain or plane stress), in Bonnet (1999) (based on 3-D symmetric Galerkin BEM) for the study of the instantaneous propagation of cracks in elastic domains. In those papers as well as in this work, the crack front is planar.

In the present formulation, it is natural to discretize the undamaged zone and the delaminated domain using triangular or quadrilateral shell elements and  $n^e$  shape functions.

As underlined in Bonnet (1999) or Ghoussoub and Leroy (2001), one should notice that the elements adjacent to the delamination front need  $C^1$  continuity along the interface between undamaged and damaged domains; so that the interpolated unit tangent  $\underline{t}$  and the normal  $\underline{y}$  are continuous along  $\Gamma_t$ . Consequently, specific elements such as those proposed in Ghoussoub and Leroy (2001) are required.

Let the nodes on  $\Gamma_t$  be numbered such that  $x^1, x^2, \dots, x^{n_f}$  are the  $n_f$  interpolation nodes along  $\Gamma_t^{\text{rupt}} \subset \Gamma_t$ . Normal velocity is supposed to be continuous versus the curvilinear coordinate along  $\Gamma_t$ . So, an admissible interpolation of  $\phi$  may be denoted:

$$\phi(s) = \sum_{i=1}^{i=n_f} N_i(\xi_t) \phi^{(i)},$$

where

- $s = s(\xi_t)$  on the side of element  $\Delta_e$  neighboring the front  $\Gamma_t^{\text{rupt}}$ ,
- the functions  $N_i(\xi_t)$  are constructed so that a  $C^1$  interpolation of the delamination front can be defined when using this form of normal velocity. This requirement is necessary in order to evaluate the actual evolution of the front due to the increment of the loading parameter with the same set of functions and thus obtain a regularized front.

Admissible propagation velocity vectors are obtained by enforcing  $\phi(s) \geq 0$  on  $\Gamma_t^{\text{rupt}}$ , which translates into inequality constraints on the normal velocity degrees of freedom  $\{\phi\} = \{\phi^{(i)}\}_{i \in \{1, \dots, n_f\}}$ .

#### A.2. Discretized equations for the rate boundary value problem

The global functional  $F$  introduced in Theorem 1 is written as:

$$F(\{V\}, \{\phi\}, \dot{\lambda}) = \frac{1}{2}^T \{V\} [K] \{V\} - \dot{\lambda}^T \{V\} \{F\} - {}^T \{\phi\} [C] \{V\} - \frac{1}{2} {}^T \{\phi\} [G^*] \{\phi\},$$

where

- $\{V\}$  is the discretized displacement velocity vector ( $n^e \times 1$ ),
- $\{\phi\}$  is the discretized normal velocity on the delamination front ( $n_f \times 1$ ),
- $[K]$  is the  $n^e \times n^e$  symmetric tangent stiffness matrix of the structure defined as

$$\frac{1}{2} {}^T \{V\} [K] \{V\} = \sum_{i=0}^2 \int_{\Omega_i} \frac{1}{2} [\dot{\boldsymbol{\epsilon}}, \dot{\boldsymbol{\kappa}}] W^{i''} \begin{bmatrix} \dot{\boldsymbol{\epsilon}} \\ \dot{\boldsymbol{\kappa}} \end{bmatrix} d\omega,$$

- $[C]$  is the coupling ( $n_f \times n^e$ )-matrix given by

$${}^T \{\phi\} [C] \{V\} = \int_{\Gamma_t} \widehat{G}(\dot{\underline{u}}, \dot{w}) c \frac{\partial g}{\partial G} ds,$$

with  $c \frac{\partial g}{\partial G} = \sum_{i=1}^{i=n_f} N_i(\xi_\Gamma) \phi^{(i)}$ ;

- $[G^*]$  is the  $n_f \times n_f$  symmetric matrix

$$\frac{1}{2} \{\phi\} [G^*] \{\phi\} = \frac{1}{2} \int_{\Gamma_t} G^* c^2 \left\{ \frac{\partial g}{\partial G} \right\}^2 ds,$$

- $\{F\}$  corresponds to the discretization of the loading (generalized forces). It is assumed that this ( $n^e \times 1$ )-vector is constant (if the mesh is unchanged) due to hypothesis of proportional loading (Section 2.6).

Consequently, the rate boundary value problem is written

$$[K] \{V\} - {}^T [C] \{\phi\} = \dot{\lambda} \{F\}.$$

The consistency relation implies that:

$$[C] \{V\} + [G^*] \{\phi\} = 0.$$

Given a normal velocity vector  $\{\phi\}$  and a loading parameter rate  $\dot{\lambda}$ , the velocity is obtained through:

$$\{V\} = [K]^{-1} \{{}^T [C] \{\phi\} + \dot{\lambda} \{F\}\}.$$

In this work, the rate boundary value problem on  $\{V\}$  is similar to the original equilibrium system which governs the displacement associated to a given loading parameter with  $\Gamma_t$  fixed. Then, the rate problem is rewritten as:

find  $\{\phi\}$  admissible with propagation criterion such that

$$[\tilde{H}] \{\phi\} + \dot{\lambda} \{H_{c-\lambda}\} = 0,$$

where  $[\tilde{H}]$  is a  $n_f \times n_f$  symmetric matrix and  $\{H_{c-\lambda}\}$  a  $n_f \times 1$  vector defined by

$$[\tilde{H}] = \{[C][K]^{-1T}[C] + [G^*]\} \quad \text{and} \quad \{H_{c-\lambda}\} = [C][K]^{-1} \{F\}.$$

### A.3. Comments

The existence and the uniqueness of the actual normal velocity defined by  $\{\phi\}$  is determined by the properties of the matrix  $[\tilde{H}]$ .

- If  $[\tilde{H}]$  is positive definite, then the uniqueness criterion is satisfied and the actual velocity is obtained, if it exists, by:

$$\{\phi\} = \dot{\lambda}[\tilde{H}]^{-1}\{H_{c-\lambda}\}.$$

- If  $[\tilde{H}]$  is not positive definite, the uniqueness criterion is violated.

To test the existence of a solution to the quasi-static instantaneous evolution, one has to investigate the properties of the quadratic function  $\frac{1}{2}^T\{\phi^*\}[\tilde{H}]\{\phi^*\}$  on admissible normal velocities.

- If it is definite positive, there exists a solution given by the previous equation.
- If it is not definite positive, then the existence or stability criterion is not fulfilled.
- If there exists a non-zero admissible velocity  $\{\phi^*\}$  such that  $[\tilde{H}]\{\phi^*\} = 0$ , the present loading parameter is a dead load.

## References

Anquez, L., Gillette de Saint-Joseph, D., Jeggy, T., Renard, J., 1990. Study of the delamination of composite bar: comparison between various computation methods. *Eur. J. Mech. A/Solids* 9 (4), 283–311.

Bonnet, M., 1999. Stability of crack fronts under Griffith criterion: a computational approach using integral equations and domain derivatives of potential energy. *Comput. Methods Appl. Mech. Eng.* 173, 337–364.

Borg, R., Nilsson, L., Simonsson, K., 2002. Modeling of delamination using a discretized cohesive zone and damage formulation. *Compos. Sci. Technol.*

Bruno, D., Greco, F., 2001. Mixed mode delamination in plates: a refined approach. *Int. J. Solids Struct.* 38 (50–51), 9149–9177.

Cochelin, B., 1988. Taux de restitution de l'énergie en théorie des poutres. *C.R. Acad. Sci. Paris, Série II* 307, 129–132.

Cochelin, B., Potier-Ferry, M., 1991. A numerical model for buckling and growth of delaminations in composites laminates. *Comput. Mech. Appl. Mech.* 89, 361–380.

Duvaut, G., Lions, J.L., 1976. Inequalities in Mechanics and Physics. Springer.

Ghoussoub, J., Leroy, Y.M., 2001. Solid–fluid phase transformation within grain boundaries during compaction by pressure solution. *J. Mech. Phys. Solids* 49, 2385–2430.

Greco, F., Lonetti, P., Zinno, R., 2002. An analytical delamination model for laminated plates including bridging effects. *Int. J. Solids Struct.* 39 (9), 2435–2463.

Hutchinson, J.W., He, M.Y., Evans, A.G., 2000. The influence of imperfections on the nucleation and propagation of buckling driven delaminations. *J. Mech. Phys. Solids* 48, 709–734.

La Saponara, V., Muliana, H., Haj-Ali, R., Kardomateas, G.A., 2002. Experimental and numerical analysis of delamination growth in double cantilever laminated beams. *Eng. Fract. Mech.* 69 (6), 687–699.

Larson, P.L., 1991. On multiple delamination buckling and growth in composite plates. *Int. J. Solids Struct.* 27 (13), 1623–1637.

Love, A.E.H., 1944. A Treatise on the Mathematical Theory of Elasticity. Dover Publications, New York.

Moon, M.-W., Chung, J.-W., Lee, K.-R., Oh, K.H., Wang, R., Evans, A.G., 2002. An experimental study of the influence of imperfections on the buckling of compressed thin films. *Acta Mater.* 50 (5), 1219–1227.

Nguyen, Q.S., 1987. Bifurcation and post-bifurcation analysis in plasticity and brittle damage. *J. Mech. Phys. Solids* 35 (3), 303–324.

Nguyen, Q.S. (Ed.), 1993. Bifurcation and Stability of Dissipative Systems. In: CISM, Courses and Lectures, vol. 327. Springer.

Nguyen, Q.S., Stolz, C., Debruyne, G., 1990. Energy methods in fracture mechanics: stability, bifurcation and second variations. *Eur. J. Mech. A/Solids* 9 (2), 157–173.

Nilsson, K.-F., Asp, L.E., Sjögren, A., 2001. On transition of delamination growth behaviour for compression loaded composite panels. *Int. J. Solids Struct.* 38 (46–47), 8407–8440.

Ousset, Y., 1999. Numerical simulation of delamination growth in layered composite plates. *Eur. J. Mech. A/Solids* 18, 291–312.

Perret, L., Allix, O., Lévèque, D., 1998. Identification and forecast of delamination in composite laminates by an interlaminar interface model. *Compos. Sci. Technol.* 58 (5), 671–678.

Petitniot, J.L., Fabis, J., 1989. Analyse expérimentale du flambement local de panneaux composites sollicités en compression. Technical Report 89/28, Institut de Mécanique des Fluides de Lille.

Pradeilles-Duval, R.-M., 1992. Evolution des systèmes avec surfaces de discontinuités mobiles: Application au délaminage. PhD thesis, Ecole Polytechnique, décembre.

Pradeilles-Duval, R.-M., 2001. Etude thermodynamique de l'évolution sous chargement dynamique de structures délamинées. *C.R. Acad. Sci. Paris, Série II* 329, 535–540.

Pradeilles-Duval, R.-M., Stolz, C., 1995. Mechanical transformation and discontinuities along a moving surface. *J. Mech. Phys. Solids* 43 (1), 91–121.

Qiu, Y., Crisfield, M.A., Alfano, G., 2001. An interface element formulation for the simulation of delamination with buckling. *Eng. Fract. Mech.* 68 (16), 1755–1776.

Roudolff, F., Ousset, Y., 2002. Comparison between two approaches for the simulation of delamination growth in a dcb specimen. *Aerospace Sci. Technol.* 6, 123–130.

Storakers, B., 1991. Nonlinear aspects of delamination in structural members. In: Proceeding XVIIth International Congress Theoretical Applied Mechanics, pp. 315–336.

Storakers, B., Anderson, B., 1988. Nonlinear plate theory applied to delamination in composites. *J. Mech. Phys. Solids* 36, 689–718.

Storakers, B., Nilsson, K.F., 1993. Imperfection sensitivity at delamination buckling and growth. *Int. J. Solids Struct.* 30, 1057–1074.

Timoshenko, S., Woinowsky-Krieger, S., 1959. Theory of Plates and Shells. McGraw-Hill Kogakusha, Librairie Polytechnique Ch. Béranger, Tokyo; Düsseldorf; London.

Zou, Z., Reid, S.R., Li, S., Soden, P.D., 2002. Application of a delamination model to laminated composite structures. *Compos. Struct.* 56 (4), 375–389.